Math 103: Measure Theory and Complex Analysis
Fall 2018

10/03/18

Lecture 9

2.) $\mu \overset{\text{Def.}}{=} \mu^o|_{\mathcal{M}^o}$ is a measure and $\mathcal{M}^o$ is a σ algebra

We first prove the claim:

Claim Let $(A_i)_{i\in\mathbb{N}} \subset \mathcal{M}^o$ be a sequence of mutually disjoint sets in $\mathcal{M}^o$. Then

$$A := \biguplus_{i\in\mathbb{N}} A_i \in \mathcal{M}^o \quad \text{and} \quad \mu^o(A) = \sum_{i\in\mathbb{N}} \mu^o(A_i).$$

proof Let $A_1, A_2 \in \mathcal{M}^o$ be the first two disjoint sets in A. Then setting $B := B \cap (A_1 \cup A_2)$ in the definiton of the measurable set A_1 we get

$$\mu^o(B \cap (A_1 \cup A_2)) = \qquad \text{for all} \quad B \subset X.$$

For the finite union $\biguplus_{i=1}^n A_i \in \mathcal{M}^o$ of disjoint sets $(A_i)_{i=1}^n \subset \mathcal{M}^o$ is in $\mathcal{M}^o$ by **part 1.c)**. By induction we have for all $n \in \mathbb{N}$

$$\mu^o\left(B \cap \biguplus_{i=1}^n A_i\right) = \qquad \text{for all} \quad B \subset X.$$

Hence by the definition of $\biguplus_{i=1}^n A_i$ as a measurable set we have for all $B \subset X$:

$$\mu^o(B) =$$

By passing to the limit this implies

$$\mu^o(B) \overset{(*)}{\geq} \qquad .$$

Here the last two inequalities follow from the countable subadditivity of μ^o:

$\sum_{i=1}^\infty \mu^o(B \cap A_i) \geq \mu^o(B \cap A)$ and as $B = (B \cap A) \cup (B \cap A^c)$. Hence in total

$$\boxed{\mu^o(B) = \mu^o(B \cap A) + \mu^o(B \cap A^c)} \quad \text{for all} \quad B \subset X.$$

This implies that A is measurable and setting $B = A$ in (*) the second part of our claim. $\square$

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3. $\mu = \mu^o|_{\mathcal{M}^o}$ **is complete**

By the definition of completeness we have to show that every subset $C \subset A \in \mathcal{M}^o$, of a set A of measure zero is measurable and has measure zero, i.e. $\mu(C) = 0$. We know that

$$\mu^o(B) = \underbrace{\mu^o(A \cap B)}_{=0 \text{ by Def. } \mu^o\text{,part b)}} + \mu^o(A^c \cap B) = \mu^o(A^c \cap B) \quad \text{for all} \quad B \subset X. \tag{1}$$

As $B = (C \cap B) \uplus (C^c \cap B)$ we know by the subadditivity of μ^o

$$\mu^o(B) \leq$$

Furthermore as $C \subset A$ we have that $C^c \cap B = A^c \cap B \uplus (C^c \cap A \cap B)$. Hence

$$\mu^o(C^c \cap B) \leq$$

In total we have that $\mu^o(C \cap B) = 0$ and

$$\mu^o(B) = \mu^o(C \cap B) + \mu^o(C^c \cap B) = \mu^o(A^c \cap B) \quad \text{for all} \quad B \subset X.$$

That means that $C \in \mathcal{M}^o$ and has measure zero. □

Chapter 2 - Special measures

Chapter 2.1 - Lebesgue measure on $\mathbb{R}$

Outline Though we have learned a lot about measures and measurable sets, we still have not defined a nice measure λ on $\mathbb{R}$, such that $\lambda([a,b]) = \ell([a,b]) = b - a$. We will do this using a corresponding outer measure λ^o. Then we will show that not all sets of $\mathcal{P}(X)$ are measurable with respect to the σ algebra Λ^o induced by λ and not all sets of Λ^o are Borel sets.

Defintion 1 (Lebesgue measure) Let I, I_k denote an open interval in $\mathbb{R}$. For $A \subset \mathbb{R}$ we set

$$\lambda^o(A) := \inf\{\sum_{n\in\mathbb{N}} \ell(I_n) \mid A \subset \bigcup_{n\in\mathbb{N}} I_n\}$$

Picture

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Proposition 2 With the definition above we have

a) λ^o is an outer measure on $\mathbb{R}$. We denote the induced **measure** $\lambda^o|_{\Lambda^o}$ on Λ^o by λ.

b) If I is an interval, then $\lambda(I) = \ell(I)$.

c) For all $a \in \mathbb{R}$ we have that $(a, +\infty) \in \Lambda^o$.

proof see H.L. Royden and P.M. Fitzpatrick, *Real Analysis, 4th edition, Chapter 2.2.*

Corollary 3 $\mathcal{B}(\mathbb{R}) \subset \Lambda^o$.

proof As $(a, +\infty) \in \Lambda^o \Rightarrow$.

□

Proposition 4 If $E \subset \mathbb{R}$ and $x \in \mathbb{R}$. Then $\lambda^o(E + x) = \lambda^o(E)$.
If $E \in \Lambda^o$ and $x \in \mathbb{R}$ then $E + x \in \Lambda$ and $\lambda(E + x) = \lambda(E)$.

proof Idea: Translated intervals have the same length. Therefore by passing to the inf we can show that $\lambda^o(E + x) = \lambda^o(E)$. The second part follows from the definition of a measurable set of an outer measure. □

Question We know three non-trivial σ algebras on $\mathbb{R}$:

$$\mathcal{B}(\mathbb{R}) \subset \Lambda^o \subset \mathcal{P}(X) \qquad \text{Are those inequalities strict?}$$

Answer Yes, the Vitali set V is a subset of $\mathcal{P}(X)$ that is not Lebesque measurable. Furthermore there is a subset A of the Cantor set C is a set that is Lebesque measurable, but not in $\mathcal{B}(\mathbb{R})$. We will construct these two examples and prove this statement.

Vitali Set

Let $X = [0, 1)$ and define

$$x \oplus y = \begin{cases} x + y & \text{if} \quad x + y < 1 \\ x + y - 1 & \phantom{\text{if}} \quad x + y \geq 1 \end{cases}.$$

This map can be seen as $X = \mathbb{R} \mod \mathbb{Z}$ and $x \oplus y = x + y \mod \mathbb{Z}$.

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Lemma 5 If $E \subset [0,1)$ is in Λ^o then $E \oplus y = \{x \oplus y \mid x \in E\} \in \Lambda^o$ for any $y \in [0,1)$. Moreover, $\lambda(E \oplus y) = \lambda(E)$.

proof For fixed $y \in [0,1)$ set

$$\Lambda^o \ni E_1 = \underbrace{E}_{\in\Lambda^o} \cap \underbrace{[0,1-y)}_{\in\Lambda^o} \quad \text{and} \quad \Lambda^o \ni E_2 = \underbrace{E}_{\in\Lambda^o} \cap \underbrace{[1-y,1)}_{\in\Lambda^o}$$

Then by the additivity of λ we have $\lambda(E) = \lambda(E_1) + \lambda(E_2)$. By construction and **Proposition 4** we have that

Hence $E \oplus y$ is measurable and again by the additivity of $\lambda : \lambda(E \oplus y) = \lambda(E)$. □

We now define an equivalence relation on $[0,1)$:

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}$$

and denote by $[x]$ the class of x.

Definition (Vitali set) Let $V \subset [0,1)$ be a complete set of representatives. This set is called a **Vitali set**. Let $(q_i)_{i=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0,1)$. with $q_0 = 0$ and set for all i

$$V_i = V \oplus q_i$$

Then

1.) If $i \neq j$ then $V_i \cap V_j = \emptyset$:

    Hence $v_j - v_i \in \mathbb{Q}$. This means that v_i and v_j are in the same equivalence class. But there is only one representative per class, hence $v_i = v_j$ and therefore $r_i = r_j$.

2.) $\biguplus_{i=0}^{\infty} V_i = [0,1)$

3.) $\lambda^o(V) = \lambda^o(V_0) = \lambda^o(V_1) = \ldots$
