

## How Many Borel Sets are There?

**Object.** This series of exercises is designed to lead to the conclusion that if  $\mathcal{B}_{\mathbf{R}}$  is the  $\sigma$ -algebra of Borel sets in  $\mathbf{R}$ , then

$$\text{Card}(\mathcal{B}_{\mathbf{R}}) = \mathfrak{c} := \text{Card}(\mathbf{R}).$$

This is the conclusion of problem 4. As a bonus, we also get some insight into the “structure” of  $\mathcal{B}_{\mathbf{R}}$  via problem 2. This just scratches the surface. If you still have an itch after all this, you want to talk to a set theorist. This treatment is based on the discussion surrounding [1, Proposition 1.23] and [2, Chap. V §10 #31].

For these problems, you will need to know a bit about well-ordered sets and transfinite induction. I suggest [1, §0.4] where transfinite induction is [1, Proposition 0.15]. Note that by [1, Proposition 0.18], there is an uncountable *well ordered* set  $\Omega$  such that for all  $x \in \Omega$ ,  $I_x := \{y \in \Omega : y < x\}$  is countable. The elements of  $\Omega$  are called the *countable ordinals*. We let  $1 := \inf \Omega$ . If  $x \in \Omega$ , then  $x + 1 := \inf\{y \in \Omega : y > x\}$  is called the *immediate successor* of  $x$ . If there is a  $z \in \Omega$  such that  $z + 1 = x$ , then  $z$  is called the *immediate predecessor* of  $x$ . If  $x$  has no immediate predecessor, then  $x$  is called a *limit ordinal*.<sup>1</sup>

1. Show that  $\text{Card}(\Omega) \leq \mathfrak{c}$ . (This follows from [1, Propositions 0.17 and 0.18]. Alternatively, you can use transfinite induction to construct an injective function  $f : \Omega \rightarrow \mathbf{R}$ .)<sup>2</sup>

2. If  $X$  is a set, let  $\mathcal{P}(X)$  be the set of subsets of  $X$  — i.e.,  $\mathcal{P}(X)$  is the *power set* of  $X$ . Let  $\mathcal{E} \subset \mathcal{P}(X)$ . The object of this problem is to give a “concrete” description of the  $\sigma$ -algebra  $\mathcal{M}(\mathcal{E})$  generated by  $\mathcal{E}$ . (Of course, we are aiming at describing the Borel sets in  $\mathbf{R}$  which are generated by the collection  $\mathcal{E}$  of open intervals.) For convenience, we assume that  $\emptyset \in \mathcal{E}$ .

Let

$$\mathcal{E}^c := \{E^c : E \in \mathcal{E}\} \quad \text{and} \quad \mathcal{E}_\sigma = \left\{ \bigcup_{i=1}^{\infty} E_i : E_i \in \mathcal{E} \right\}.$$

(Note, I just mean that  $\mathcal{E}_\sigma$  is the set of sets formed from countable unions of elements of  $\mathcal{E}$ . Since  $\emptyset \in \mathcal{E}$ ,  $\mathcal{E} \subset \mathcal{E}_\sigma$ .)

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<sup>1</sup>The set of countable ordinals has a rich structure. We let  $2 := 1 + 1$ , and so on. The set  $\{n \in \mathbf{N}\} \subset \Omega$  is countable, and so has a supremum  $\omega$  (see [1, Proposition 0.19]). Then there are ordinals  $\omega + 1, \omega + 2, \dots, 2\omega, 2\omega + 1, \dots, \omega^2, \omega^2 + 1, \dots, \omega^\omega$ , and so on.

<sup>2</sup>The issue of whether or not  $\text{Card}(\Omega) = \mathfrak{c}$  is the *continuum hypothesis*, and so is independent of the usual (ZFC) axioms of set theory.

We let  $\mathcal{F}_1 := \mathcal{E} \cup \mathcal{E}^c$ . If  $x \in \Omega$ , and if  $x$  has an immediate predecessor  $y$ , then we set

$$\mathcal{F}_x := (\mathcal{F}_y)_\sigma \cup ((\mathcal{F}_y)_\sigma)^c.$$

If  $x$  is a limit ordinal, then we set

$$\mathcal{F}_x := \bigcup_{y < x} \mathcal{F}_y.$$

We aim to show that

$$\mathcal{M}(\mathcal{E}) = \bigcup_{x \in \Omega} \mathcal{F}_x \tag{†}$$

- (a) Observe that  $\mathcal{F}_1 \subset \mathcal{M}(\mathcal{E})$ .
- (b) Show that if  $\mathcal{F}_y \subset \mathcal{M}(\mathcal{E})$  for all  $y < x$ , then  $\mathcal{F}_x \subset \mathcal{M}(\mathcal{E})$ . Then use transfinite induction to conclude that  $\mathcal{F}_x \subset \mathcal{M}(\mathcal{E})$  for all  $x \in \Omega$ .
- (c) Show that the right-hand side of (†) is closed under countable unions.
- (d) Conclude that  $\bigcup_{x \in \Omega} \mathcal{F}_x$  is a  $\sigma$ -algebra, and that (†) holds.

3. Recall that if  $A$  and  $B$  are sets, then  $\prod_{a \in A} B$  is simply the set of functions from  $A$  to  $B$ . For reasons that are unclear to me, this set is usually written  $B^A$ . Notice that  $\prod_{i=1}^{\infty} B = \prod_{i \in \mathbf{N}} B$  is just the collection of sequences in  $B$ . Notice also that  $\text{Card}(B^A)$  depends only on  $\text{Card}(A)$  and  $\text{Card}(B)$ .

- (a) Check that

$$\prod_{i=1}^{\infty} \left( \prod_{j=1}^{\infty} B \right) = \prod_{(i,j) \in \mathbf{N} \times \mathbf{N}} B. \tag{*}$$

Thus the cardinality of either side of (\*) is the same as  $\prod_{i=1}^{\infty} B$ .

- (b) Use these observations together with the fact that  $\text{Card}(\prod_{i=1}^{\infty} \{0, 1\}) = \mathfrak{c} := \text{Card}(\mathbf{R})$  (which follows from [1, Proposition 0.12]) to show that

$$\text{Card}\left(\prod_{i=1}^{\infty} \mathbf{R}\right) = \mathfrak{c}.$$

- (c) Show that if  $\text{Card}(\mathcal{E}) = \mathfrak{c}$ , then  $\text{Card}(\mathcal{E}_\sigma) = \mathfrak{c}$ .

4. Let  $\mathcal{B}_{\mathbf{R}}$  be the  $\sigma$ -algebra of Borel sets in  $\mathbf{R}$ . In [1, Proposition 0.14(b)], it is shown that if  $\text{Card}(A) \leq \mathfrak{c}$  and if  $\text{Card}(Y_x) \leq \mathfrak{c}$  for all  $x \in A$ , then  $\bigcup_{x \in A} Y_x$  has cardinality bounded by  $\mathfrak{c}$ . By following the given steps, use this observation, as well as problems 2 and 3, to show that

$$\text{Card}(\mathcal{B}_{\mathbf{R}}) = \mathfrak{c}. \quad (\ddagger)$$

- (a) Let  $\mathcal{E}$  be the collection of open intervals (including the empty set) in  $\mathbf{R}$ . Then  $\text{Card}(\mathcal{E}) = \mathfrak{c}$ .
- (b)  $\mathcal{B}_{\mathbf{R}} = \mathcal{M}(\mathcal{E})$ .
- (c) Define  $\mathcal{F}_x$  as in problem 2. Use transfinite induction and problem 3 to prove that  $\text{Card}(F_x) = \mathfrak{c}$  for all  $x \in \Omega$ .
- (d) Use problem 2 to conclude that  $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\mathbf{R}}$  has the cardinality claimed in  $(\ddagger)$ .<sup>4</sup>

## References

- [1] Gerald B. Folland, *Real analysis*, Second, John Wiley & Sons Inc., New York, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.
- [2] Anthony W. Knap, *Basic real analysis*, Cornerstones, Birkhäuser Boston Inc., Boston, MA, 2005. Along with a companion volume *Advanced real analysis*.

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<sup>4</sup>It is my *understanding* that the classes  $\mathcal{F}_x$  are all distinct; that is,  $\mathcal{F}_x \subsetneq \mathcal{F}_y$  if  $x < y$  in  $\Omega$ . But I don't have a reference or a proof at hand.