Cartoon Counting and the Cantor Set

1. The Cantor-Lebesgue Function

The Cantor Set: Let $C_0 = [0,1]$. Let $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ be the closed set obtained by removing the open middle third of C_0. In general, let C_n the union of 2^n disjoint closed intervals obtained by removing the open middle thirds of each of the closed intervals in C_{n-1}. Then the Cantor set is

$$\mathscr{C} = \bigcap_{n=1}^{\infty} C_n.$$

Ternary Expansions: If $\alpha \in [0,1]$, then we can find $\big(\alpha_k\big)_3 \in \prod_{k=1}^{\infty} \{\,0,1,2\,\}$ such that

$$\alpha = \sum_{k=1}^{\infty} \frac{\alpha_k}{3^k} \tag{1}$$

and conversely every $\big(\alpha_k\big)_3 \in \prod_{k=1}^{\infty} \{\,0,1,2\,\}$ represents an $\alpha \in [0,1]$ via (1). In the world of cartoon characters (with three fingers), we might write $\alpha = 0.\alpha_1\alpha_2\alpha_3\cdots$. Unfortunately, we notice that sometimes the expansion in (1) is not unique. For example, $\frac{4}{9} = 0.11 = 0.102222\ldots$. However the expansion is only not unique when α is of the form $p3^{-n}$. Then it has finite expansion—that is, one with $\alpha_k = 0$ for all $k > n$, and another with $\alpha_k = 2$ for all $k > n$. In $(p,3) = 1$, then one of these expansions will have $a_n = 1$ and the other will have a_n equal to 0 or 2. Suppose that when we have two such expansions, we always choose the one with a_n equal to 0 or 2. For example,

$$\frac{1}{3} = 0.022\ldots \quad \frac{2}{3} = 0.2 \quad \frac{1}{9} = 0.0022\ldots \quad \frac{2}{9} = 0.02.$$

If we do this, then $\alpha_1 =$ exactly when $\frac{1}{3} < \alpha < \frac{2}{3}$. Similarly, if $\alpha_1 \neq 1$ and $\alpha_2 = 1$ then we have exactly $\frac{1}{9} < \alpha < \frac{2}{9}$ or $\frac{7}{9} < \alpha < \frac{8}{9}$. With this convention, $\alpha \sim \big(\alpha_k\big)_3$ is in one of the open intervals deleted to get C_n exactly when $\alpha_n = 1$.

Definition 1. Suppose that $\alpha \in [0,1]$ has the ternary expansion $\big(\alpha_k\big)_3$. Then let

$$N = \begin{cases} \infty & \alpha_k \neq 1 \text{ for all } k\text{, and} \\ k & \text{if } \alpha_k = 1 \text{ and } \alpha_j \neq 1 \text{ for } j < k. \end{cases}$$

Then the *Cantor-Lebesgue Function* is the function $\phi : [0,1] \to [0,1]$ given by

$$\phi(\alpha) = \sum_{k=1}^{N} \frac{b_k}{2^k} \tag{2}$$

where

$$b_k = \begin{cases} 0 & \text{if } \alpha_k = 0, \\ 1 & \text{if } \alpha_k = 2, \text{ and} \\ 1 & \text{if } N < \infty \text{ and } k = N. \end{cases}$$

Remark 2. It is comforting to note that our definition of ϕ in (2) does not depend on any choice of expansion for α as the exact same phenomena occurs in the binary expansion.

Lemma 3. *The Cantor-Lebesgue function* $\phi : [0,1] \to [0,1]$ *is a continuous nondecreasing surjection. Furthermore,* $\phi(\mathscr{C}) = [0,1]$ *and* $\phi'(\alpha) = 0$ *for all* $\alpha \in [0,1] \setminus \mathscr{C}$*.*

Proof. Since every $\alpha \in [0,1]$ also has a binary expansion, it is clear that ϕ is surjective. In fact, since $\alpha \in \mathscr{C}$ if and only if α has a ternary expansion $\big(\alpha_k\big)_3$ with each $\alpha_k \in \{0,2\}$, it is clear that $\phi(\mathscr{C}) = [0,1]$. Moreover, it follow from the preceding discussion that ϕ is constant on the closure each of the deleted open intervals. Moreover, $\alpha, \beta \in \mathscr{C}$ and $\alpha < \beta$ then $\phi(\alpha) < \phi(\beta)$ unless α and β are the endpoints of the same deleted open interval.

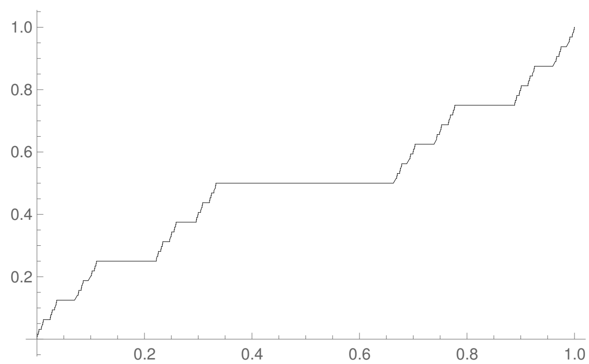

FIGURE 1. The Graph of the Cantor-Lebesgue Function

In follows that ϕ is nondecreasing—see Figure 1.[1] Since it maps onto $[0,1]$, it can have no jump discontinuities. Hence it must be continuous. Since $[0,1] \setminus \mathscr{C}$ is open and ϕ is constant on the open intervals making up $[0,1] \setminus \mathscr{C}$, we certainly have $\phi'(x) = 0$ there. □

Recall that a topological space is *totally disconnected* if its only connected subsets are points—that is, its connected components are reduced to points.

[1] I pulled this graph from the Wolfram's web site `https://mathworld.wolfram.com/CantorFunction.html` where one can access the `Mathematica` source.

Corollary 4. *The Cantor set $\mathscr{C}$ is a totally disconnected compact subset of Lebesgue measure zero which is uncountable and has the same cardinality $\mathfrak{c}$ as that of the continuium.*

Proof. We have $m(C_n) = 2^2 \cdot \frac{1}{3^n}$ and $m(\mathscr{C}) = \lim_n m(C_n)$. Since $\phi(\mathscr{C}) = [0,1]$, we must have $\operatorname{Card}(\mathscr{C}) = \mathfrak{c}$. To see that $\mathscr{C}$ is totally disconnected, it suffices to see that given $\alpha < \beta$ in $\mathscr{C}$ there is a $\gamma \in [0,1] \setminus \mathscr{C}$ such that $\alpha < \gamma < \beta$. But we can take n large enough so that the closed intervals in C_n have length less than $\frac{r}{3}$. Then α and β are contained in disjoint intervals $[a,b]$ and $[c,d]$ with $b < c$. Then either $(b,c) \in O := [0,1] \setminus \mathscr{C}$ or O contains closed interval I in some C_m with $m \geq n$. In the first case, any $\gamma \in (b,c)$ will do. In the second, we can take γ in the middle third of I.[2] □

Now consider the function $f : [0,1] \to [0,2]$ defined by

$$f(x) = x + \phi(x).$$

Then f is trivially continuous and hence surjective (by the Intermediate-Value Theorem). If

$$f(x) = x + \phi(x) = y + \phi(y) = f(y),$$

then

$$x - y = \phi(y) - \phi(x).$$

Since ϕ is nondecreasing, this forces $x = y$. Hence f is a continuous bijection between to compact Hausdorff sets and must be a homeomorphism. Since ϕ is constant on each open interval in $[0,1] \setminus \mathscr{C}$, f maps each such interval onto an interval of equal length. Therefore $m\big(f([0,1]) \setminus \mathscr{C}\big) = 1$. Therefore $m\big(f(\mathscr{C})\big) = 1$. It follows that there is nonmeasurable set $P \subset f(\mathscr{C})$. Let $A = f^{-1}(P)$. Since $A \subset \mathscr{C}$ and $m(\mathscr{C}) = 0$, A is Lebesgue measurable. If we let $g = f^{-1}$, then g is continuous and $g^{-1}(A) = f(A) = P$. Since P is not measurable, A can't be a Borel set. (Recall that since g is continuous, it is Borel and hence measurable from $([0,2], \mathcal{B}([0,2]))$ to $([0,1], \mathcal{B}([0,1]))$.)

Now we have given a proper proof of the following.

Theorem 5. *The Lebesgue measurable sets contain the Borel sets as a proper subset.*

[2]This also shows that $\mathscr{C}$ has no isolated points. Then the Baire Category Theorem implies $\mathscr{C}$ is uncountable. However we would have to invoke the Continuium Hypothesis to conclude from that alone that $\operatorname{Card}(\mathscr{C}) = \mathfrak{c}$.

But there is a darker mystery unveiled here as well. Let $h = \mathbb{1}_A$. Then $h : ([0,1], \mathcal{L}[0,1]) \to \mathbf{R}$ is Lebesgue measurable as in the continuous function g. But

$$\begin{aligned}(h \circ g)^{-1}\big((\tfrac{1}{2}, \infty\big) &= g^{-1}\big(h^{-1}(\tfrac{1}{2}, \infty)\big)\\ &= g^{-1}(A)\\ &= P.\end{aligned}$$

Since P is not measurable, neither is the composition $h \circ g$ of a Lebesgue measurable function h and the continuous—hence Lebesgue measurable—function g.[3]

Lemma 6. *The composition $h \circ g$ of Lebesgue measurable functions need not measurable—even if g is continuous.*

While this observation is a bit disconcerting at first blush, it is actually not surprising if we look at these maps as maps between measurable spaces. In our example above, we have

$$\begin{array}{ccccc} \big([0,2], \mathcal{B}([0,2])\big) & \xrightarrow{\;g\;} & \big([0,1], \mathcal{B}([0,1])\big) & & \\ & & \Big\downarrow \text{id} & & \\ & & \big([0,1], \mathcal{L}[0,1]\big) & \xrightarrow{\;h\;} & \big(\mathbf{R}, \mathcal{B}(\mathbf{R})\big). \end{array}$$

Since $\mathcal{L}[0,1]$ properly contains $\mathcal{B}([0,1])$, the identity map $\text{id} : \big([0,1], \mathcal{B}([0,1])\big) \to \big([0,1], \mathcal{L}[0,1]\big)$ is not measurable.

2. Classical Absolute Continuity

In the good old days when "real analysis" meant studying real-valued functions on the real line, a function $f : [a,b] \to \mathbf{R}$ was called *absolutely continuous* if for all $\epsilon > 0$ there is a $\delta > 0$ such that given finitely many open intervals $\{\, (a_k, b_k) \,\}_{k=1}^{n}$ in (a,b) such that $\sum_{k=1}^{n} (b_k - a_k) < \delta$, then $\sum_{k=1}^{n} \big|f(b_k) - f(a_k)\big| < \epsilon$. Notice that since this holds for $n = 1$, an absolutely continuous function on $[a,b]$ is necessarily (uniformly) continuous on $[a,b]$.

In [1, Theorem 6.5.10] for example, it is proven that if f is absolutely continuous on $[a,b]$, then $f'(x)$ exists for almost all $x \in (a,b)$ and we can recover f via the integral:

$$f(x) = f(a) + \int_a^x f'(t)\, dm(t).$$

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[3]Recall that the composition of a continuous function with a measurable function is always measurable.

Furthermore it is shown in [1, Theorems 6.5.11 & 6.5.14] that the converse holds: if $g \in \mathcal{L}^1([a,b])$, then

$$f(x) = c + \int_a^x g(t)\,dm(t)$$

is absolutely continuous on $[a,b]$ and $f'(x) = g(x)$ for almost all $x \in (a,b)$.

Example 7. The Cantor-Lebesgue function is an example of an almost everywhere differentiable continuous function that is not absolutely continuous. There is a proof of this given in the first example in [1, §6.5]. However, since ϕ' is zero almost everywhere, ϕ is not integral of it's derivative: the friendly formula

$$\phi(x) = \phi(0) + \int_0^x \phi'(t)\,dt = 0$$

holds only when $x = 0$ (since the intergral is always zero). This also shows that ϕ can't be absolutely continuous.

The question which presents itself is "does the classic notion of absolute continuity have anything to do with the absolute continuity of measures and the Raydon-Nikodym derivative?" Here is a simple minded answer.

Suppose that ν is a finite measure on $\mathbf{R}$ and let

$$f(x) = \nu((-\infty, x)).$$

Then f is nondecreasing. This immediately implies that f is continuous except for countably many jump discontinuities. Even better, "Lebesgue's Theorem" implies that f is differentiable almost everywhere [1, §6.2].

Suppose that $\nu \ll m$. Let $g = \frac{d\nu}{dm}$ so that

$$f(x) = \int_{-\infty}^x g(t)\,dm(t).$$

It follows that f is absolutely continuous and $f' = g$ almost everywhere.

Now suppose that f is absolutely continuous. I claim that $\nu \ll m$. For this it suffices to see that if $E \subset (a,b)$ has Lebesgue measure zero, then $\nu(E) = 0$. Fix $\epsilon > 0$. Then by definition, since f is non-decreasing, there is a $\delta > 0$ such that or any finite collection $\{\,(a_k, b_k)\,\}_{k=1}^n$ of open intervals in (a,b) we have

$$\sum_{k=1}^n (b_k - a_k) < \delta \quad \text{implies} \quad \sum_{k=1}^n f(b_k) - f(a_k) = \sum_{k=1}^n \nu([a_k, b_k)) < \epsilon.$$

Since $m(E) = 0$, we can find a countable cover $\{ (a_k, b_k) \}_{k=1}^{\infty}$ of E in (a, b) such that

$$\sum_{k=1}^{\infty} (b_k - a_k) < \delta.$$

Hence we have

$$\nu(E) \le \sum_{k=1}^{\infty} \nu((a_k, b_k)) \le \sum_{k=1}^{\infty} \nu([a_k, b_k)) = \lim_{n\to\infty} \sum_{k=1}^{n} \nu([a_k, b_k)) \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $\nu(E) = 0$ as required.

References

[1] Halsey L. Royden and Patrick M. Fitzpatrick, *Real analysis*, 4th Ed., Prentice Hall, New York, 2010.