Math 73/103: Fall 2020
Lecture 1

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The Course — Administration

- We should be recording.
- With your help, all lectures and discussions should be recorded. (See the first item!) These lectures and discussions will be available via zoom links on the webpage.
- The current model is that lectures will be recorded live during our regularly schedule class period: MWF 2:35 – 3:40. We will reserve the x-hour (Th 1:40 – 2:30) for office hours. This may change during the term to pre-recorded lectures that you should watch prior to the class period.
- You should review the material on our web page (`math.dartmouth.edu/~m103f20/`). In particular, homework will be assigned and distributed there.
- Homework and exams will be handed in via `gradescope` which I hope we can both figure out prior to the first assignment.

The Course — Syllabus

- This course has been "updated" to reflect some changes to our graduate requirements. In particular, in spite of the title of the course, there is no longer a complex analysis component. (It is possible that in 2022, Math 113 will be complex analysis. This academic year, Math 113 will be functional analysis.)
- Instead, we will start with the basics of metric spaces influenced by my prejudice in favor of functional analysis. While not technically required, some prior experience with metric spaces—as in Math 63 for example—will be very useful.
- After 3 – 4 weeks of metric space analysis, we will switch to measure theory. Some experience with the Riemann integral will be very helpful.
- Our approach to measure theory will be quite abstract, but our primary example and motivation will be the Lebesgue integral on the real line which extends the Riemann integral on closed bounded subsets.

Break One

- Just so the zoom video files are not too big, I plan to take breaks from time to time.
- This is one of those breaks.
- Everyone loses points if no one reminds me to start recording again!
- Before I stop the recording, does anyone have questions or comments?

Let's Get Started

Definition

A metric on a set X is a function $\rho : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ we have

1. [definiteness] $\rho(x, y) = 0$ implies $x = y$,
2. [symmetry] $\rho(x, y) = \rho(y, x)$ and
3. [triangle inequality] $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

Notice that item (3) implies that $\rho(x, x) = 0$ for all $x \in X$. Hence item (1) could be replaced by

1. [definiteness] $\rho(x, y) = 0$ if and only if $x = y$

without any harm. The pair (X, ρ) is called a metric space.
A function $\rho : X \times X \to [0, \infty)$ satisfying only items (2) and (3) is called a pusedo-metric. Then (X, ρ) is called a pusedo-metric space. Note that $\rho(x, x) = 0$ for all x for any pusedo-metric.

Norms

Example

1. The model example is the standard metric on $X = \mathbf{R}$ or $X = \mathbf{C}$ where $\rho(x, y) = |x - y|$.
2. More generally, if X is a vector space over $\mathbf{F}$ where $\mathbf{F}$ is either $\mathbf{R}$ or $\mathbf{C}$, then we call $\|\cdot\| : X \to [0, \infty)$ a norm on X if
    1. [definiteness] $\|x\| = 0$ if and only if $x = 0$,
    2. [homogeneity] $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbf{F}$ and $x \in X$, and
    3. [triangle inequality] $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

    If $(X, \|\cdot\|)$ is a normed vector space over $\mathbf{F}$ as above, then $\rho(x, y) = \|x - y\|$ is a metric on X called the associated metric on X.
3. Of course, $\|x\| = |x|$ is a norm on $\mathbf{F}$ viewed as a vector space over itself.

The ℓ^p Spaces

Definition

Let X be a set and $\ell^\infty(X)$ the vector space of all bounded complex-valued functions on X. Then

$$\|f\|_\infty := \sup_{x \in X} |f(x)|$$

is called the supremum norm.

Of course, you have to prove that $\|\cdot\|_\infty$ is actually a norm. An important special case occurs when $X = \mathbf{N} := \{1, 2, 3, \dots\}$. Then we write ℓ^∞ in place of $\ell^\infty(\mathbf{N})$ and view ℓ^∞ as the set of sequences in $\mathbf{C}$. Then ℓ^∞ is a normed space with the supremum norm $\|x\| = \sup_{n \in \mathbf{N}} |x_n|$.

Definition

If $1 \leq p < \infty$, then we let $\ell^p := \ell^p(\mathbf{N})$ be the set of complex valued sequences—that is, complex valued functions on $\mathbf{N}$—such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty \quad \text{(We say that } x \text{ is } p\text{-summable.)}$$

Norms on ℓ^p

Definition

If x is any sequence, then we define $\|x\|_p = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{\frac{1}{p}} \in [0, \infty]$.

1. It is not hard to check that each $1 \leq p < \infty$, ℓ^p is a complex vector space, and that $\|\cdot\|_1$ is a norm on ℓ^1. (This means you should check!)
2. Naturally, we want to know whether $\|\cdot\|_p$ is a norm on ℓ^p if $1 < p < \infty$.
3. It is clearly definite, and it is homogeneous thanks to the annoying $\frac{1}{p}$ power—that is why it is there.
4. The fact that the triangle inequality holds—so that $\|\cdot\|_p$ is actually a norm—is not so easy to establish. This result, namely that

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p,$$

   is often called Minkowski's inequality.
5. We will establish Minkowski's inequality via some homework exercises.
6. Once we have Minkowski, then it follows that $\|\cdot\|_p$ is a norm on ℓ^p for all $1 \leq p \leq \infty$.
7. Of course, there is no harm in replacing $\mathbf{C}$ by $\mathbf{R}$. Then the set $\ell^p_{\mathbf{R}}$ of p-summable real-valued sequences is a normed real vector space.

Example

Since we can view $\mathbf{C}^n = \{ (z_1, \dots, z_n) : z_k \in \mathbf{C} \}$ as a subset of ℓ^p in the obvious way, we get norms

$$\|z\|_p = \Big(\sum_{k=1}^{n} |z_k|^p \Big)^{\frac{1}{p}}$$

on $\mathbf{C}^n$ for $1 \leq p < \infty$ with the obvious counterparts on $\mathbf{R}^n$.

Example

When I teach our complex variables course, I define the complex number field $\mathbf{C}$ as $\mathbf{R}^2$ where $(x, y) \in \mathbf{R}^2 \leftrightarrow x + iy \in \mathbf{C}$. This identifies $(\mathbf{C}, |\cdot|)$ with $(\mathbf{R}^2, \|\cdot\|_2)$ as metric spaces.

More Generally

Definition

If (X, ρ) and (Y, σ) are metric spaces, then a function $f : X \to Y$ such that $\sigma\big(f(x), f(y)\big) = \rho(x, y)$ for all $x, y \in X$ is called an isometry. We say that (X, ρ) and (Y, σ) are isometric metric spaces if there is a surjective isometry from X onto Y.

Remark

Check that an isometry is necessarily injective. Hence if $f : X \to Y$ is a surjective isometry, then it is bijective and $f^{-1} : Y \to X$ is also a surjective isometry. Hence the definition of isometric metric spaces is symmetric and defines an equivalence relation on the class of metric spaces. Note that $(\mathbf{R}^2, \| \cdot \|_2)$ and $(\mathbf{C}, | \cdot |)$ are isometric with respect to their underlying metrics.

The Discrete Metric

Here is a much different sort of example.

Definition

Let X be any nonempty set. Then

$$\rho(x,y) = \begin{cases} 1 & \text{of } x \neq y \text{, and} \\ 0 & \text{if } x = y \end{cases}$$

is called the discrete metric on X.

In a lower level course, we would prove that the discrete metric is a metric. But I expect you do that on your own.

- It is time for another break.
- Before we stop the recording, does anyone have questions or comments?

Loose Ends

- We should be recording.

Definition (New metric spaces from old)

If (X, ρ) is a metric space and $Y \subset X$ is a subset of X, then the restriction of ρ to $Y \times Y$ is called the subspace metric on Y.

It is immediate that $\rho|_{Y \times Y}$ is a metric on Y and $(Y, \rho|_{Y \times Y})$ is called a (metric) subspace of X.

Example

Let $C([a, b])$ be the set of continuous functions on the closed interval $[a, b] \subset \mathbf{R}$. Then $C([a, b])$ is a subset of $\ell^\infty([a, b])$ and becomes a metric subspace with respect to $\rho(f, g) = \|f - g\|_\infty = \sup_{t \in [a,b]} |f(t) - g(t)|$.

Remark

There is nothing terribly significant about noticing that ρ is a metric on $C([a, b])$. But later we will see that sometimes subspaces inherit nice properties from the ambient space. The point of the above example will be that $C([a, b])$ can inherit such properties from $\ell^\infty([a, b])$.

Equivalent Metrics

Definition

Suppose that ρ and σ are metrics on X. Then we say that ρ and σ are strongly equivalent if there are constants $c, d > 0$ such that

$$c \cdot \rho(x, y) \leq \sigma(x, y) \leq d \cdot \rho(x, y) \quad \text{for all } x, y \in X. \quad (1)$$

Remark

Notice that if (1) holds, then since c and d are nonzero,

$$\frac{1}{d} \cdot \sigma(x, y) \leq \rho(x, y) \leq \frac{1}{c} \cdot \sigma(x, y) \quad \text{for all } x, y \in X.$$

Hence the definition of strongly equivalent metrics above is symmetric in ρ and σ.

Remark

It is a homework exercise to show that the metrics induced by the norms $\| \cdot \|_p$ with $1 \leq p \leq \infty$ are all strongly equivalent on $\mathbf{R}^n$.

An Important Example

Example

Suppose that (X, ρ) is a metric space. Then

$$d(x, y) := \frac{\rho(x, y)}{1 + \rho(x, y)}$$

is a metric on X.

Proof.

This is not so obvious! Clearly, if $d(x, y) = 0$, then $\rho(x, y) = 0$ and hence $x = y$, and d is clearly symmetric since ρ is. The triangle inequality is the tricky bit.

Let $f : [0, \infty) \to \mathbf{R}$ be given by $f(t) = \dfrac{t}{1+t}$. Then

$$f'(t) = \frac{(1 + t) - t}{(1 + t)^2} = \frac{1}{(1 + t)^2} > 0.$$

Therefore f is strictly increasing. □

Proof Continued

Continued.

Since $\rho(x,z) \le \rho(x,y) + \rho(y,z)$ and since f is increasing, we have

$$\begin{aligned} d(x,z) &= \frac{\rho(x,z)}{1+\rho(x,z)} = f(\rho(x,z)) \\ &\le f\big(\rho(x,y)+\rho(y,z)\big) \\ &= \frac{\rho(x,y)+\rho(y,z)}{1+\rho(x,y)+\rho(y,z)} \\ &\le \frac{\rho(x,y)}{1+\rho(x,y)+\rho(y,z)} + \frac{\rho(y,z)}{1+\rho(x,y)+\rho(y,z)} \\ &\le d(x,y)+d(y,z). \end{aligned}$$

□

Definition

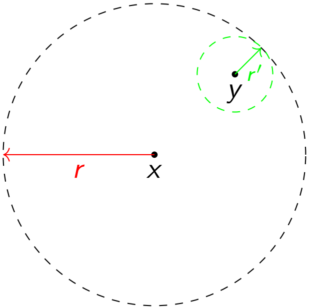
Let (X, ρ) be a metric space.

1. If $x \in X$ and $r > 0$, then
$$B_r(x) = B_r^\rho(x) = \{\, y \in X : \rho(x, y) < r \,\}$$
is called the open ball of radius r centered at x.
2. A subset $U \subset X$ is called open if for all $x \in U$ there is a $r > 0$ such that $B_r(x) \subset U$.
3. We call $D \subset X$ a neighborhood of x if there is an open set U such that $x \in U \subset D$.

Proof by Terminology

Lemma

Open balls are open.



Proof.

Suppose that $y \in B_r(x)$. Then $r' = r - \rho(x, y) > 0$. Now it will suffice to see that $B_{r'}(y) \subset B_r(x)$.
So suppose that $z \in B_{r'}(y)$. Then

$$\begin{aligned} \rho(x, z) &\leq \rho(x, y) + \rho(y, z) \\ &< \rho(x, y) + r' = r. \quad \square \end{aligned}$$

Open Sets

Proposition

Let (X, ρ) be a metric space.

1. *Both $\emptyset$ and X are open.*
2. *If U and V are open, then so is $U \cap V$.*
3. *If U_a is open for each $a \in A$, then $\bigcup_{a \in A} U_a$ is open.*

Proof.

(1) This is almost tautologous.
(2) Suppose that $x \in U \cap V$. There there are $r_k > 0$ such that $B_{r_1}(x) \subset U$ and $B_{r_2}(x) \subset V$. Then $r = \min\{ r_1, r_2 \} > 0$, and $B_r(x) \subset U \cap V$.
(3) This is straightforward. □

Topology

Definition (For those who know)

If X is a set, then we write $\mathcal{P}(X)$ for the set of all subsets of X. An element $\tau \in \mathcal{P}(X)$ is called a topology on X if

1. $\emptyset, X \in \tau$,
2. $U, V \in \tau$ implies $U \cap V \in \tau$, and
3. $U_a \in \tau$ for all $a \in A$ implies $\bigcup_{a\in A} U_a \in \tau$.

Remark

The only point here is that collection τ_ρ of open sets in a metric space (X, ρ) form a topology on X called the metric topology for ρ. I won't be able to resist using this terminology, so I thought I would make it formal. With few exceptions, the only topologies we'll work with in Math 73/103 will be metric topologies.

That's Enough for Day One

- That is enough for now.