Math 73/103: Fall 2020
Lecture 10

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Getting Started

- We should be recording!
- This a good time to ask questions about the previous lecture, complain, or tell a story.
- Speaking of complaining, let's have homework problems 11 to 23 due Wednesday via `gradescope`.

Some Review

Definition

We say that a bounded real-valued function f on $[a, b]$ is Riemann integrable if the upper Riemann integral $\overline{\mathcal{R}}\int_a^b f$ and the lower Riemann integral $\underline{\mathcal{R}}\int_a^b f$ are equal. In that case, we call the common value is denoted by $\mathcal{R}\int_a^b f$. The set of all Riemann integrable functions on $[a, b]$ is denoted by $\mathcal{R}[a, b]$.

Homework

Proposition

A bounded real-valued function on $[a, b]$ is Riemann integrable if and only if for all $\epsilon > 0$ there is a partition P of $[a, b]$ such that

$$\mathcal{U}(f, P) - \mathcal{L}(f, P) < \epsilon.$$

Proof.

I will leave this as a guided homework problem. □

Proposition

If $f : [a, b] \to \mathbf{R}$ is continuous, then $f \in \mathcal{R}[a, b]$.

Proof.

Let $\epsilon > 0$. Since $[a, b]$ is compact, f is uniformly continuous. Let $\delta > 0$ be such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/(b - a)$. Let $n \in \mathbf{N}$ be such that $\frac{b-a}{n} < \delta$. Let P_n be the regular partition $\{ t_k \}_{k=0}^{n}$ such that $t_k = a + k\frac{(b-a)}{n}$ for $0 \leq k \leq n$. By the Extreme Value Theorem, there are $c, d \in [t_{k-1}, t_k]$ such that $M_k = f(c)$ and $m_k = f(d)$. Since $|c - d| < \delta$, we have $M_k - m_k < \frac{\epsilon}{b-a}$.

Proof

Proof Continued.

Now we compute

$$\begin{aligned}\mathcal{U}(f,P)-\mathcal{L}(f,P) &= \sum_{k=1}^{n}(M_k-m_k)\Delta t_k \\ &< \frac{\epsilon}{b-a}\sum_{k=1}^{n}\Delta t_k \\ &= \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that f is Riemann integrable by our homework problem. □

Measure Zero

Definition

A subset $A \subset [a, b]$ has content zero (or later measure zero) if for all $\epsilon > 0$ there are open intervals $\{ I_n \}_{n=1}^{\infty}$ (some of which could be empty) such that

1. $A \subset \bigcup_{n=1}^{\infty} I_n$ and
2. $\sum_{n=1}^{\infty} \ell(I_n) < \epsilon$,

where $\ell\big((a, b)\big) := b - a$.

Examples

Example

1. Single points have content zero.
2. You will prove for homework that the countable union of sets of content zero have content zero.
3. In particular, countable subsets of $[a, b]$ such as $\mathbf{Q} \cap [a, b]$, have content zero.

We will, I hope, have a proof of the following later in the term using our methods, but a classical proof is available from Goldberg, *Methods of Real Analysis*, §7.3 or Knapp, *Real Analysis*, Theorem III.3.29.

Theorem

A bounded function $f : [a, b] \to \mathbf{R}$ is Riemann integrable if and only if the set A of discontinuities of f has content zero.

Examples

Example

Let $f = \mathbb{1}_{\mathbf{Q}\cap[0,1]}$ be the characteristic function of the rationals in $[0,1]$.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \cap [0,1] \text{ and} \\ 0 & \text{if } x \in [0,1] \setminus \mathbf{Q}. \end{cases}$$

Then f is nowhere continuous. Hence f is not Riemann integrable.

The Ruler Function

Example

Let $g : \mathbf{R} \to \mathbf{R}$ be the "ruler function"

$$g(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ with } (p,q) = 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It is a fun exercise to first figure out why g is called the "ruler function". One of the reasons its interesting is that it is continuous at every irrational number but discontinuous at every rational. As a result the restriction of g to any closed bounded interval $[a,b]$ is Riemann integrable. Furthermore,

$$\mathcal{R}\int_a^b g = 0$$

for any interval $[a,b]$.

Back to Riemann's Question

Remark (Problems with the Riemann Theory)

1. *Restricted to bounded functions.*
2. *Restricted to bounded intervals.*
3. *Does not have strong convergence properties.*
4. *The set $\mathcal{R}[a,b]$ does not have good properties. In particular, it does not admit a useful complete metric. Certainly, the metric $\rho(f,g) = \int_a^b |f = g|$ is not complete on $\mathcal{R}[a,b]$. (Ok, ρ is just a pseudo metric, but even after forming the appropriate equivalence relation, we don't get a complete metric.)*

Frankly, items (1) and (2) are just minor limitations, while (3) is still just an inconvenience. But (4) is truly a problem. If we have learned nothing else from the beginning of this term, it is—to paraphrase the motto of Faber College—that "completeness is good".

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Riemann Sums

Remark

For motivation for what comes next, let's also dredge up the concept of a Riemann sum.

Definition

Let $P = \{ t_k \}_{k=0}^n$ be a partition of $[a, b]$. Choose $\xi_k \in [t_{k-1}, t_k]$ and let $\xi = (\xi_1, \dots, \xi_n)$. Then

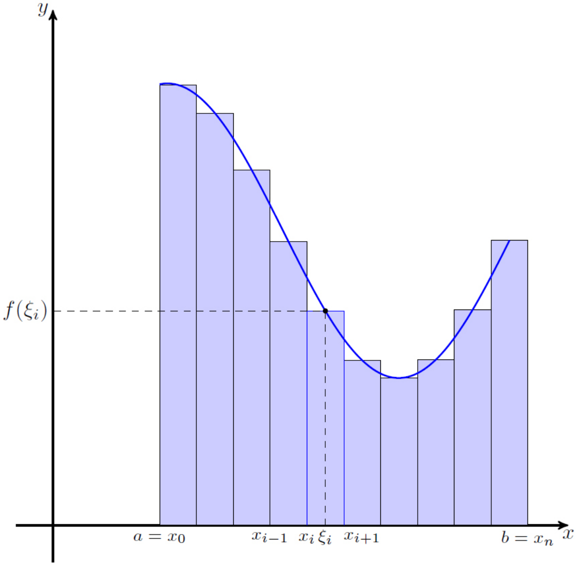
$$\mathcal{R}(f, P, \xi) := \sum_{k=1}^n f(\xi_i)\Delta t_k = \sum_{k=1}^n f(\xi_i)\ell([t_{k-1}, t_k])$$

is called a Riemann sum for f. ▸ Return ▸ Return We call

$$\|P\| = \max_{1 \le k \le n} \Delta t_k = \max_{1 \le k \le n} \ell([t_{k-1}, t_k])$$

the mesh of the partition P.

Picture



Remark

The following is the way Riemann integrability is presented in most of the basic calculus courses we teach. It takes a bit of hard work to see that it is equivalent to the "lower/upper sum" version of the definition we've settled on here. I am quoting it here primarily for motivation, although you can use it on homework if you like.

Theorem (Knapp, *Real Analysis*, Theorem III.3.27)

Suppose that f is a bounded real-valued function on $[a, b]$. Then $f \in \mathcal{R}[a, b]$ if and only if there is a $I \in \mathbf{R}$ such that for all $\epsilon > 0$, there is a $\delta > 0$ such that for all partitions P with $\|P\| < \delta$ we have

$$\left|\mathcal{R}(f, P, \xi) - I\right| < \epsilon$$

provided $\xi_k \in [t_{k-1}, t_k]$. Moreover, when $f \in \mathcal{R}[a, b]$, then $I = \mathcal{R}\int_a^b f$.

Motivation

Remark

Crudely put, the ideal of the Lebesgue integral is allow ▸ Riemann sums *that allow partition of* $[a, b]$ *into more general subsets—rather than just intervals. However, this requires that we assign a "length" to these subsets that is compatible with our notion of the length* $\ell(I)$ *of and interval. This suggests that we want a function* $m : \mathcal{P}([0, 1]) \to [0, \infty)$ *such that*

1. $m(\emptyset) = 0$,
2. $m(I) = \ell(I)$ *when* I *is an interval, and*
3. *If* $E = \bigcup_{n=1}^{\infty} E_n$ *with* $E_n \cap E_m = \emptyset$ *if* $n \neq m$, *then*

$$m(E) = \sum_{n=1}^{\infty} m(E_n).$$

Remark

Unfortunately, asking for a function m as on the previous slide is mired in a set-theoretic morass even working in ZFC. With mild translation invariant requirements, it is not possible to define such a m on all subsets of $\mathbf{R}$—we will make this precise in due course. One solution would be replace item (3) with finite disjoint unions. But it turns out that this leads to a less useful theory. The accepted "solution" is to restrict our generalized length function m to a subset $\mathcal{M} \subset \mathcal{P}([a, b])$. Even so, we need $\mathcal{M}$ to be robust. It should contain all intervals. We also want it closed under countable set operations like intersection and union. Then we can only consider functions $f : [a, b] \to \mathbf{R}$ that "play nice" with $\mathcal{M}$. For example, we want $f^{-1}(I) \in \mathcal{M}$ for any interval $I \subset \mathbf{R}$.

Making an Integral

Suppose $f : [a, b] \to [0, \infty)$ plays nice with $\mathcal{M}$ as on the previous slide. If f takes only finite many values $\alpha_1, \dots, \alpha_n \in \mathbf{R} \setminus \{ 0 \}$, then we are assuming

$$A_k = f^{-1}(\alpha_k) = \{ x \in [a, b] : f(x) = \alpha_k \} \in \mathcal{M}.$$

Then we define

$$\operatorname{Int}(f) = \sum_{k=1}^{n} \alpha_k m(A_k)$$

which should remind you of a ▸ Riemann sum. We call such a function a simple function. For a general $f : [a, b] \to [0, \infty)$, we define

$$\int_{[a,b]} f(x)\, dm(x) = \sup\{ \operatorname{Int}(s) : s \text{ is a simple function and } 0 \le s \le f \}.$$

Break Time

- Definitely time for a break.
- Questions?
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Getting Technical

- Now that we see what the idea is, it is time to start getting precise.
- Strictly for motivation, let's review a bit of topology.

Definition

Recall that a topology on any set X is a collection $\tau \subset \mathcal{P}(X)$ such that

1. $\emptyset, X \in \tau$,
2. τ is closed under finite intersections, and
3. τ is closed under arbitrary unions.

Furthermore, a function $f : (X, \tau) \to (Y, \sigma)$ is continuous if $f^{-1}(V) \in \tau$ for all $V \in \sigma$.

Remark

Of course, if life were fair, you would only have to know about topologies that arise from metrics. But life is not fair.

σ-Algebras

Definition

A collection $\mathcal{M} \subset \mathcal{P}(X)$ is called a σ-algebra in X if

1. $X \in \mathcal{M}$,
2. $A \in M$ implies $A^C := X \setminus A \in \mathcal{M}$, and
3. $\{ A_n \}_{n=1}^{\infty} \subset \mathcal{M}$ implies that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

If $\mathcal{M}$ is a σ-algebra in X, then the pair $(X, \mathcal{M})$ is called a measurable space.

Bonus Properties

Remark

Suppose that $\mathcal{M}$ is a σ-algebra in X.

1. $\emptyset \in \mathcal{M}$.
2. *If* $A_1, \dots, A_n \in \mathcal{M}$, *then* $A_1 \cup \cdots \cup A_n \in \mathcal{M}$. *(Just let* $A_k = \emptyset$ *if* $k > n$*.)*
3. *If* $\{ A_n \}_{n=1}^{\infty} \subset \mathcal{M}$, *then* $\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^C \right)^C \in \mathcal{M}$.
4. *If* $A_1, \dots, A_n \in \mathcal{M}$, *then* $A_1 \cap \cdots \cap A_n \in \mathcal{M}$. *(Just let* $A_k = X$ *if* $k > n$ *or take complements.)*
5. *If* $A, B \in \mathcal{M}$, *then* $A \setminus B = A \cap B^C \in \mathcal{M}$.

Low Hanging Fruit

Example

1. $\mathcal{M} = \{ \emptyset, X \}$.
2. $\mathcal{M} = \mathcal{P}(X)$.
3. Suppose that X is uncountable. Let $\mathcal{M}$ be the collection of $A \subset X$ such that either A or A^C is countable. Then you can check, yes you really should, that $\mathcal{M}$ is a σ-algebra in X.

Remark

If you find the above examples less than stimulating, you are just normal. As it happens, producing "reasonable" non-trivial examples of σ-algebras is hard.

Generating σ-Algebras

Proposition

Let X be a set and $\mathscr{N} \subset \mathcal{P}(X)$ any subset. Then there is a smallest σ-algebra $\mathcal{M}(\mathscr{N})$ in X such that $\mathscr{N} \subset M(\mathscr{N})$. We call $\mathcal{M}(\mathscr{N})$ the σ-algebra generated by $\mathscr{N}$.

Proof.

Let $\mathfrak{S}$ be the set of σ-algebras $\mathcal{M}'$ in X such that $\mathscr{N} \subset \mathcal{M}'$. Note that $\mathcal{M}' = \mathcal{P}(X) \in \mathfrak{S}$, so $\mathfrak{S} \neq \emptyset$. Let

$$\mathcal{M} = \bigcap_{\mathcal{M}' \in \mathfrak{S}} \mathcal{M}'.$$

Once you check that $\mathcal{M}$ is a σ-algebra, then it is clear that $\mathcal{M}$ is the smallest σ-algebra containing $\mathcal{N}$. Therefore we can let $\mathcal{M}(\mathscr{N}) = \mathcal{M}$. □

Definition

Let (X, τ) be a topological space. Then the Borel σ-algebra in X is the σ-algebra $\mathcal{B}(X)$, generated by τ. (That is, $\mathcal{B}(X) = \mathcal{M}(\tau)$.) The elements of $\mathcal{B}(X)$ are called Borel sets.

Non-Trivial Example?

Remark

It is still not clear that we have exhibited a non-trivial σ*-algebra! There is no reason that* $\mathcal{B}(X)$ *can't be all of* $\mathcal{P}(X)$*. For example, if we give* X *the discrete metric—that is,* $\rho(x, y) = 1 - \delta_{xy}$*—then every subset is open and* $\mathcal{B}(X) = \mathcal{P}(X)$*. We will eventually see that* $\mathcal{B}(\mathbf{R}) \neq \mathcal{P}(\mathbf{R})$*, but this is non-trivial! Worse, the structure of* $\mathcal{B}(\mathbf{R})$ *is very subtle. Obviously,* $\mathcal{B}(\mathbf{R})$ *contains the open sets. But it also contains countable intersections of open sets called* G_δ *sets. But it contains countable unions of* G_δ *sets called* $G_{\delta\sigma}$*-sets. Then* $G_{\delta\sigma\delta}$*-sets, and so on. Alternatively, we could start with closed sets, then form countable unions called* F_σ*-sets. Then* $F_{\sigma\delta}$*-sets,* $F_{\sigma\delta\sigma}$*-sets, and so on. Fascinatingly, each of these clases is distinct and do not exhaust the collection of Borel sets! One needs to continue transfinitely using all countable ordinals. Fortunately, almost none of this need concern the practicing analyst.*

That's Enough for Today

- That is enough for now.