Math 73/103: Fall 2020
Lecture 11

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Getting Started

- We should be recording!
- This a good time to ask questions about the previous lecture, complain, or tell a story.
- Homework problems 11 to 23 are due today via `gradescope`.

Measusrable Functions

Definition

If $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is a function between measurable spaces, then we say that f is measurable if $f^{-1}(A) \in \mathcal{M}$ for all $A \in \mathcal{N}$.

Definition (Unfortunate Convention)

If $(X, \mathcal{M})$ is a measurable space and (Y, τ) is a topological space, then we say that $f : X \to Y$ is measurable if $f : (X, \mathcal{M}) \to (Y, \mathcal{B}(Y))$ is measurable.

Proposition

If Y is a topological space, then $f : X \to Y$ is measurable if and only if $f^{-1}(V) \in \mathcal{M}$ for all open subsets $V \subset Y$.

Proof

Proof.

Suppose that $f : (X, \mathcal{M}) \to (Y, \mathcal{B}(Y))$ is measurable. Then if $V \in \tau$, $V \in \mathcal{B}(Y)$ and $f^{-1}(V) \in \mathcal{M}$ by assumption.

Now suppose that $f^{-1}(V) \in \mathcal{M}$ for all $V \in \tau$. Let $\mathcal{N} = \{ A \subset Y : f^{-1}(A) \in \mathcal{M} \}$. But assumption, $\tau \subset \mathcal{N}$. Therefore it will suffice to see that $\mathcal{N}$ is a σ-algebra in Y. Since $f^{-1}(Y) = X \in \mathcal{M}$, we have $Y \in \mathcal{N}$. Now if $A \in \mathcal{N}$, then $f^{-1}(A) \in \mathcal{M}$. Hence $f^{-1}(A)^C = f^{-1}(A^C) \in \mathcal{M}$ and $A^C \in \mathcal{N}$. Now suppose that $\{ A_n \}_{n=1}^{\infty} \subset \mathcal{N}$. Then $f^{-1}(A_n) \in \mathcal{M}$. Hence $\bigcup f^{-1}(A_n) = f^{-1}\Big(\bigcup A_n\Big) \in \mathcal{M}$. Thus $\bigcup A_n \in \mathcal{N}$. Therefore $\mathcal{N}$ is a σ-algebra in Y as required. □

Borel Functions

Definition

If Y and Z are topological spaces, then $f : Y \to Z$ is Borel if $f^{-1}(V) \in \mathcal{B}(Y)$ for all open sets V in Z.

Proposition

Suppose that Y and Z are topological spaces.

1. *$f : Y \to Z$ is Borel if and only if $f : (Y, \mathcal{B}(Y)) \to (Z, \mathcal{B}(Z))$ is measurable.*
2. *If $f : Y \to Z$ is continuous, then f is Borel.*
3. *If $f : Y \to Z$ is Borel and $g : (X, \mathcal{M}) \to Y$ is measurable, then $f \circ g : X \to Z$ is measurable.*

Proof

Proof.

(1) and (2) are straightforward and left for you to check. As for (3), note that if $V \subset Z$ is open, then

$$(f \circ g)^{-1}(V) = g^{-1}\bigl(f^{-1}(V)\bigr).$$

Since f is Borel, $f^{-1}(V) \in \mathcal{B}(Y)$. But then by definition, $g^{-1}\bigl(f^{-1}(V)\bigr) \in \mathcal{M}$. □

Generalize

Proposition

Suppose that $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ *and* $g : (Y, \mathcal{N}) \to (Z, \mathcal{P})$ *are measurable. Then* $g \circ f : (X, \mathcal{M}) \to (Z, \mathcal{P})$ *is measurable.*

Proof.

Just like part (3) on the previous slide. □

A Basis for a Topology

Definition

If (Y, τ) is a topological space, then $\beta \subset \tau$ is called a basis for τ if every element of τ is a union of elements from β.

Lemma

Let (Y, τ) be a topological space. Then β is a basis for the topology if given $U \in \tau$ and $x \in U$ there is a $V \in \beta$ such that $x \in V \subset U$.

Proof.

For each $x \in U$ let $V_x \in \beta$ be such that $x \in V_x \subset U$. Then $U = \bigcup_{x \in U} V_x$. □

Examples

Example

1. If (X, ρ) is a metric space then the collection of all open balls is a basis. In fact, we could take all open balls of rational radius or even of radii $\frac{1}{n}$ for $n \in \mathbf{N}$.
2. The collection of open intervals with rational endpoints form a basis for the usual topology on $\mathbf{R}$.

Definition

A topological space Y is said to be second countable if there is a countable basis for the topology on Y.

Proposition

A separable metric space is second countable.

Proof

Proof.

Let $D = \{ d_n \}_{n=1}^{\infty}$ be a countable dense subset of (X, ρ). Let

$$\beta = \{ B_{\frac{1}{n}}(d_m); n, m \in \mathbf{N} \}.$$

Then $\beta \subset \tau_\rho$. Suppose $V \in \tau_\rho$ and $x \in V$. Then there is a $r > 0$ such that $B_r(x) \subset V$. Let n be such that $\frac{1}{n} < \frac{r}{2}$. Since D is dense, there is a $d_m \in D$ such that $d_m \in B_{\frac{1}{n}}(x)$. If $y \in B_{\frac{1}{n}}(d_m)$, then

$$\rho(y, x) \leq \rho(y, d_m) + \rho(d_m, x) < \frac{1}{n} + \frac{1}{n} < r.$$

That is, $x \in B_{\frac{1}{m}}(d_m) \subset B_r(x) \subset V$.

□

Second Countability

Corollary

$\mathbf{R}^n$ *is second countable for all* $n \geq 1$ *as is* ℓ^p *for* $1 \leq p < \infty$.

Proposition

Suppose that Y *is a second countable topological space and that* β *is a countable basis for the topology on* Y. *Then* $f : (X, \mathcal{M}) \to Y$ *is measurable if and only if* $f^{-1}(V) \in \mathcal{M}$ *for all* $V \in \beta$.

Proof.

By definition, every open set is a union of elements of β. Since β is countable, every open set is a countable union of elements of β. Etc. □

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Functions on $\mathbf{R}^2$

Proposition

Let Y be a topological space. Suppose that $u, v : (X, \mathcal{M}) \to \mathbf{R}$ are measurable, and that $\varphi : \mathbf{R}^2 \to Y$ is continuous. Then

$$h(x) = \varphi\big(u(x), v(x)\big)$$

is measurable from $(X, \mathcal{M})$ to Y.

Remark

For the proof we need to know that $\mathbf{R}^2$, it its usual topology, has a countable basis β of sets of the form $A \times B$ with A and B open in $\mathbf{R}$. This is not hard to prove using our lemma characterizing a basis for a topology proved earlier.

Proof

Proof.

I claim that $f(x) = \big(u(x), v(x)\big)$ is a measurable function from $(X, \mathcal{M})$ to $(\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2))$. To verify the claim, it suffices by the previous remark to see that $f^{-1}(A \times B) \in \mathcal{M}$ if A, B are open in $\mathbf{R}$. But $f^{-1}(A \times B) = u^{-1}(A) \cap v^{-1}(B) \in \mathcal{M}$ as required. This proves the claim.

Now $h = \varphi \circ f$, and since φ is continuous, we have already seen that this composition is measurable:

$$(X, \mathcal{M}) \xrightarrow{\ f\ } (\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2)) \xrightarrow{\ \varphi\ } (Y, \tau).$$

□

Low Hanging Fruit

Corollary

Suppose that $f, g : (X, \mathcal{M}) \to \mathbf{R}$ *are measurable. Then so are* $f \pm g$ *and* fg. *Further,* $h = f + ig$ *is measurable from* $(X, \mathcal{M})$ *to* $\mathbf{C}$.

Proof.

Note that $\varphi_1(x, y) = x \pm y$ and $\varphi_2(x, y) = xy$ are continuous from $\mathbf{R}^2 \to \mathbf{R}$ while $\varphi_3(x, y) = x + iy$ is continuous from $\mathbf{R}^2 \to \mathbf{C}$. □

More Fruit

Corollary

If $f, g : (X, \mathcal{M}) \to \mathbf{C}$ are measurable, then so are $|f|$, $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, $f \pm g$, and fg.

Proof.

Then maps $z \mapsto |z|$, $z \mapsto \mathrm{Re}(z)$, and $z \mapsto \mathrm{Im}(z)$ are all continuous from $\mathbf{C} \to \mathbf{R}$. Then recall that the composition of a continuous function with a measurable function is measurable.

Combining with the previous corollary, we see that the real and imaginary parts of $f \pm g$ and fg are measurable. The second part of that corollary then implies $f \pm g$ and fg are measurable. □

Supremums

Remark

Let (f_n) be a sequence of real-valued functions. Then we will want to consider $g(x) = \sup_n f_n(x)$. But this is problematical: the supremum might be $+\infty$!

Definition

The extended real numbers $[-\infty, \infty]$ is the topological space $\mathbf{R} \cup \{ \pm\infty \}$ with a basis consisting of the sets $[-\infty, b)$, (a, b), and $(a, \infty]$ for all $a, b \in \mathbf{R}$. We give $[-\infty, \infty]$ its natural order.

Remark

Of course, we can take $a, b \in \mathbf{Q}$ above so that $[-\infty, \infty]$ is second countable. In fact, $[-\infty, \infty]$ is homeomorphic to $[-1, 1]$ in an order preserving way. So we could pull the metric back from $[-1, 1]$ and view $[-\infty, \infty]$ as a compact metric space.

Measurable Functions into $[-\infty, \infty]$

Lemma

A function $f : (X, \mathcal{M}) \to [-\infty, \infty]$ is measurable if and only if

$$\{\, x \in X : f(x) > a \,\} = f^{-1}((a, \infty]) \in \mathcal{M} \quad \textit{for all } a \in \mathbf{R}. \tag{1}$$

Proof.

If f is measurable, then (1) must be measurable. So the only interesting bit is to assume that (1) holds and verify that f is measurable. Since $[-\infty, \infty]$ is second countable, it suffices to consider our basis.

Proof

Proof Continued.

Note that $f^{-1}([-\infty, a]) = f^{-1}((a, \infty])^C \in \mathcal{M}$. But

$$[-\infty, a) = \bigcup_{n=1}^{\infty} [-\infty, a - \frac{1}{n}].$$

And then

$$f^{-1}([-\infty, a)) = \bigcup_{n=1}^{\infty} f^{-1}([-\infty, a - \frac{1}{n}]) \in \mathcal{M}.$$

However, then

$$f^{-1}((a, b)) = f^{-1}((a, \infty]) \cap f^{-1}([-\infty, b)) \in \mathcal{M}$$

as well. This completes the proof. □

Break Time

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"Recall"

- Let (a_n) be a sequence in $[-\infty, \infty]$.
- Let $b_k = \sup_{n \geq k} a_n$. (Note that $b_k \in [-\infty, \infty]$.)
- Since $b_{k+1} \leq b_k$, $\lim_{k \to \infty} b_k = \inf_k b_k$ always exists in $[-\infty, \infty]$.
- We define $\limsup_n a_n = \lim_{k \to \infty} \sup_{n \geq k} a_n$.
- Similarly, we define $\liminf_n a_n = \lim_{k \to \infty} \inf_{n \geq k} a_n = \sup_k \inf_{n \geq k} a_n$.
- Part of the reason for working in $[-\infty, \infty]$ is that then the $\limsup a_n$ and $\liminf a_n$ always exist.

Proposition

Let (a_n) be a sequence in $[-\infty, \infty]$. Then

$$\liminf_n a_n \le \limsup_n a_n.$$

Furthermore, $\lim_n a_n$ *exists in* $[-\infty, \infty]$ *if and only if* $\liminf_n a_n = \limsup_n a_n$, *and then* $\lim_n a_n$ *is the common value.*

Proof.

This is what your parents paid all that college tuition for. □

That's Enough for Today

- That is enough for now.