Math 73/103: Fall 2020
Lecture 12

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Getting Started

- We should be recording!
- As usual, this a good time to ask questions about the previous lecture, complain, or tell a story.
- I've graded and returned your second homework via `gradescope`. I've also updated the solutions on the assignments page. Everyone should carefully review the solutions.
- Note that I don't always look at every problem. For problems that I don't grade, I just check to see if it was attempted.
- Please bring questions and concerns to office hours.
- Keep in mind, that the goal is not just a correct solution, but an elegant write-up as well.

Theorem

Suppose that $f_n : (X, \mathcal{M}) \to [-\infty, \infty]$ *is measurable for all* $n \in \mathbf{N}$. *Then so are the following:*

1. $g = \sup_{n \geq 1} f_n$.
2. $h = \limsup_n f_n$.
3. $k = \inf_{k \geq 1} f_n$.
4. $r = \liminf_n f_n$.

Proof.

Let $g(x) = \sup_{n\geq 1} f_n(x)$. Then if $g(x) > \alpha$, then there is a n such that $f_n(x) > \alpha$. Since the other direction is immediate, it follows that $g^{-1}\big((a,\infty]\big) = \bigcup_{n=1}^{\infty} f_n^{-1}\big((a,\infty]\big)$. Therefore g is measurable.

But $k = \inf_{n\geq 1} f_n = -\sup_{n\geq 1} -f_n$. Hence k is measurable.

Now $h = \limsup_n f_n = \inf_k \big(\sup_{n\geq k} f_n\big)$, so h is measurable.

Then $r = \liminf_n f_n = -\limsup_n -f_n$, so r is measurable. □

Pointwise Limits

Corollary (Pointwise limits of measurable functions are measurable)

Suppose that $Y = [-\infty, \infty]$ *or* $Y = \mathbf{C}$. *Suppose also that* $f_n : (X, \mathcal{M}) \to Y$ *is measurable for all* $n \in \mathbf{N}$, *and that* $\lim_{n\to\infty} f_n(x) = f(x)$ *for all* $x \in X$. *Then* $f : (X, \mathcal{M}) \to Y$ *is measurable.*

Proof.

If $Y = [-\infty, \infty]$, then $f = \limsup_n f_n$ and f is measurable. If $Y = \mathbf{C}$, then $\operatorname{Re}(f) = \lim_n \operatorname{Re}(f_n)$ while $\operatorname{Im}(f) = \lim_n \operatorname{Im}(f_n)$. Hence the real and imaginary parts of f are measurable. Therefore f is measurable. □

Back To Riemann's Ghost

Remark

We can already see an improvement over the Riemann theory here. Let $\{ r_n \}_{n=1}^{\infty}$ *be an* enumeration *of the rationals in* $[0,1]$*. Let* $A_n = \{ r_1, r_2, \dots, r_n \}$ *and let* $f_n = \mathbb{1}_{A_n}$*. Then* $f_n \in \mathcal{R}[0,1]$*. But* (f_n) *converges pointwise to* $f = \mathbb{1}_{[0,1]\cap\mathbf{Q}}$ *which is not Riemann integrable! (Well, assuming here that* $[0,1]$ *does not have content zero. Although we will prove this in due course, it is not so hard to show.)*

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Definition

A measure on a measurable space $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$, and
2. (countable additivity) If $\{ A_n \}_{n=1}^{\infty} \subset \mathcal{M}$ and if $A_n \cap A_m = \emptyset$ if $m \neq n$, then
$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

The triple $(X, \mathcal{M}, \mu)$ is called a measure space.

Low Hanging Fruit

Proposition

Suppose that $(X, \mathcal{M}, \mu)$ *is a measure space.*

1. *(monotonicity) If* $A, B \in \mathcal{M}$ *and* $A \subset B$*, then* $\mu(A) \leq \mu(B)$.
2. *In part (1), if* $\mu(A) < \infty$*, then* $\mu(B \setminus A) = \mu(B) - \mu(A)$.
3. *If* $A_n \in \mathcal{M}$ *and* $A_n \subset A_{n+1}$ *for all* $n \in \mathbf{N}$*, then* $\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \mu(A_n)$.
4. *If* $A_n \in \mathcal{M}$ *and* $A_{n+1} \subset A_n$ *for all* $n \in \mathbf{N}$*, and if* $\mu(A_1) < \infty$*, then* $\mu\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_n \mu(A_n)$.

return

Proof

Proof.

(1) $B = A \cup B \setminus A$ is a disjoint union. Hence $\mu(B) = \mu(A) + \mu(B \setminus A)$.

(2) If $\mu(A) < \infty$, then we can subtract it from both sides.

(3) This is similar. Let $B_1 = A_1$, and let $B_n = A_n \setminus A_{n-1}$ if $n \geq 2$. Then $\{ B_n \}_{n=1}^{\infty}$ are pairwise disjoint. Moreover, $\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k = A_n$. Hence

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \mu\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} \mu(B_n)$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n\to\infty} \mu\Big(\bigcup_{k=1}^{n} B_k\Big)$$
$$= \lim_{n\to\infty} \mu(A_n).$$

Proof

Proof Continued.

(4) Let $C_n = A_1 \setminus A_n$. Then $C_n \subset C_{n+1}$ and $\infty > \mu(A_1) \leq \mu(A_n) \leq \mu\left(\bigcap_{n=1}^{\infty} A_n\right)$. Therefore on the one hand,

$$\mu\left(\bigcup C_n\right) = \mu\left(A_1 \setminus \bigcap A_n\right) = \mu(A_1) - \mu\left(\bigcap A_n\right).$$

On the other hand,

$$\begin{aligned}\mu\left(\bigcup C_n\right) &= \lim_{n\to\infty} \mu(C_n) = \lim_{n\to\infty} \mu(A_1) - \mu(A_n) \\ &= \mu(A_1) - \lim_{n\to\infty} \mu(A_n).\end{aligned}$$

Since $\mu(A_1) < \infty$, the result follows. □

Example (Homework Problem #30)

Let X be a set and let $\mathcal{M} = \mathcal{P}(X)$.

1. Define
$$\nu(E) = \begin{cases} \infty & \text{if } E \text{ is infinite, and} \\ |E| & \text{if } E \text{ is finite} \end{cases}$$
where $|E|$ is the number of elements in E. This is the measure obtained by taking $f(x) = 1$ for all $x \in X$ in problem #30. The measure ν is called counting measure on X.

2. Pick any $x_0 \in X$. Let
$$\delta_{x_0}(E) = \begin{cases} 1 & \text{if } x_0 \in E \text{, and} \\ 0 & \text{if } x_0 \notin E. \end{cases}$$
This is the measure obtained by taking $f = \mathbb{1}_{\{x_0\}}$. It is called the Dirac measure at x_0.

More Examples

Example

Suppose that X is uncountable and that
$\mathcal{M} = \{ E \subset X : \text{either } E \text{ or } E^C \text{ is countable} \}$. Then

$$\rho(E) = \begin{cases} 1 & \text{if } E \text{ is uncountable, or} \\ 0 & \text{if } E \text{ is countable.} \end{cases}$$

You'll show that ρ is a measure on homework.

Example (Non-trivial)

We will devote a lot of effort to proving that there is a unique measure m on $\mathcal{B}(\mathbf{R})$ such that $m(I) = \ell(I)$ for every interval and $m(E + x) = m(E)$ for all $E \in \mathcal{B}(\mathbf{R})$ where $E + x = \{ y + x : y \in E \}$. At this point, it is worth noting that we don't even know that $E + x \in \mathcal{B}(\mathbf{R})$ if $E \in \mathcal{B}(\mathbf{R})$.

An Example

Example

Let ν be counting measure on $\mathbf{N}$. Let $A_n = \{\, n, n+1, n+2, \dots \,\}$. Then $A_{n+1} \subset A_n$. But

$$0 = \nu(\emptyset) = \nu\Big(\bigcap_{n=1}^{\infty} A_n\Big) \neq \lim_{n\to\infty} \nu(A_n) = \infty.$$

Thus the hypothesis that $\nu(A_1) < \infty$ is necessary in the nested intersection result from earlier.

Break Time

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Simple Functions

Definition

A function $f : X \to \mathbf{C}$ is called simple if $s(X)$ is finite. We say that s is a non-negative simple function (hereafter NNSF) if $s(X) \subset [0, \infty)$. (Note that the value ∞ is excluded—there is no such element in $\mathbf{C}$!)

Remark

If $s(X) \setminus \{0\} = \{\, \alpha_1, \ldots, \alpha_n \,\}$, and if $A_k = s^{-1}(\alpha_k)$, then it is not hard to work out that s is measurable if and only if each $A_k \in \mathcal{M}$. (Note that $s^{-1}(0) = (A_1 \cup \cdots \cup A_n)^C$.) Then

$$s = \sum_{k=1}^{n} \alpha_k \mathbb{1}_{A_k}.$$

This representation of s as a linear combination of characteristic functions is unique if we assume the α_k are distinct and non-zero.

Approximation by Simple Functions

Theorem

Suppose that $f : (X, \mathcal{M}) \to [0, \infty]$ is any function. Then there are NNSFs, s_k, on X such that

1. $0 \le s_1 \le s_2 \le \cdots \le f$, *and such that*
2. *for all* $x \in X$, $\lim_{n\to\infty} s_n(x) = f(x)$.

Furthermore, if f is measurable, we can assume each s_k is a measurable non-negative simple function (hereafter MNNSF). If f is bounded, then $s_n \to f$ uniformly on X.

Remark

Since the pointwise limit of measurable functions is always measurable, the theorem above implies that a non-negative extended real-valued function f is measurable if and only if there are MNNSFs, s_n, such that $s_n \nearrow f$ pointwise.

Proof

Proof.

Let $n \in \mathbf{N}$. For each $x \in [0, \infty)$, let $k_n(x)$ be the unique $k \in \mathbf{Z}$ (with $k \geq 0$) such that

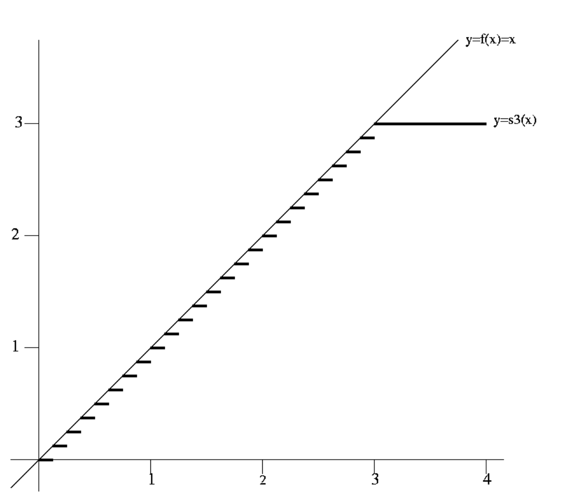
$$2^{-n}k \leq x < 2^{-n}(k+1).$$

Then define $\varphi_n : [0, \infty] \to [0, \infty)$ by

$$\varphi_n(x) = \begin{cases} k_n(x)2^{-n} & \text{if } 0 \leq x \leq n, \text{ and} \\ n & \text{if } n \leq x \leq \infty. \end{cases}$$

Proof Continued.

Here is a picture of the graph of $y = \varphi_3(x)$ I stole from the web:



Proof

Proof Continued.

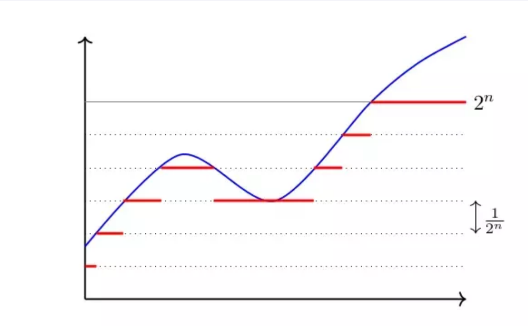
Notice that each φ_n is a Borel function on $[0,\infty)$ and $0 \le \varphi_1 \le \varphi_2 \le \cdots \le x$. Moreover, for all $x \in [0,n]$,

$$x - 2^{-n} \le \varphi_n(x) \le x.$$

It follows that $\varphi_n(x) \to x$ on $[0,\infty]$ and that the convergence is uniform on any interval $[0,N]$ with $N \in \mathbf{N}$.

Proof Continued.

We now let $s_n = \varphi_n \circ f$. Again, I stole a picture of a sample graph of such a s_n from the web. It is poorly labeled—you should replace 2^n on the right by n, but I hope this gives you an idea:



Proof

Proof Continued.

The point is that, since φ_n is Borel, s_n is measurable if f is. In any case, we have $s_n(x) \to f(x)$ for all $x \in X$. If f is bounded—say $0 \leq f(x) \leq N$, then it follows from our construction of the φ_n that the convergence is uniform. □

That's Enough for Today

- That is enough for now.