Math 73/103: Fall 2020
Lecture 13

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Getting Started

- We should be recording!
- As usual, this a good time to ask questions about the previous lecture, complain, or tell a story.
- Our next homework will be due on the 21st. We'll see just what problems when we see where we are Friday.

Remark (Slesnick's Ghost)

When I was a new Dartmouth faculty member, the legendary Bill Slesnick drummed it in to those of us who wanted to rise to the level of a true Dartmouth mathematics professor that "infinity is not a number". In particular, limits "diverge to infinity"—they do not "converge to infinity". Nevertheless, $\pm\infty$ are perfectly good points in $[-\infty, \infty]$. However, Bill would be delighted to point out that addition in $[-\infty, \infty]$ is not everywhere defined: e.g., $\infty - \infty$ does not make sense. But in $[0, \infty]$ we can make do—with all due apologies to the great Professor Slesnick. We agree that $0 \cdot \infty = 0$, that $a + \infty = \infty$ for all $a \in [0, \infty]$, and $a \cdot \infty = \infty$ if $a > 0$. However, $a + b = a + c$ only implies $b = c$ if $a \in [0, \infty)$ and $ab = ac$ only implies $b = c$ if $a \in (0, \infty)$. One important property that we do have is that if $a_n \nearrow a$ and $b_n \nearrow b$, then $a_n b_n \nearrow ab$! A first example of the use of this later property is the next lemma.

Extended Non-Negative Real-Valued Functions

Lemma

Suppose that $f, g : (X, \mathcal{M}) \to [0, \infty]$ are measurable. Then so are $f + g$ and fg.

Proof.

Choose MNNSFs $s_n \nearrow f$ and $t_n \nearrow g$. Then $s_n + t_n \nearrow f + g$ and $s_n t_n \nearrow fg$. This suffices as we have already proved that the pointwise limit of measurable functions is measurable. □

Simple Integrals

Definition

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $s : X \to [0, \infty)$ is a MNNSF. Let $s(X) \setminus \{0\} = \{ \alpha_1, \dots, \alpha_n \}$ with the α_k distinct. Then

$$s = \sum_{k=1}^{n} \alpha_k \mathbb{1}_{A_k} \tag{$*$}$$

is called the standard representation of s. (Note that $A_k = s^{-1}(\alpha_k)$ is measurable in this case.) Then for all $E \in \mathcal{M}$ we define

$$\operatorname{Int}_E(s) = \sum_{k=1}^{n} \alpha_k \cdot \mu(A_k \cap E). \tag{$\dagger$}$$

Remark

The point of taking the standard representation of s in $(*)$ is to insure that $(\dagger)$ is well defined. Also our convention that $0 \cdot \infty = 0$ eliminates any concern over omitting 0 from $\{ \alpha_1, \dots, \alpha_n \}$.

Definition

Suppose that $f : (X, \mathcal{M}) \to [0, \infty]$ is measurable. If $E \in \mathcal{M}$, then we define

$$\int_E f \, d\mu = \int_E f(x) \, d\mu(x) := \sup_{0 \leq s \leq f} \mathrm{Int}_E(s)$$

where the supremum is taken over all MNNSFs dominated by f.

What is the Deal with the Standard Representation?

Lemma

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $\{ B_j \}_{j=1}^{m} \subset \mathcal{M}$ are pairwise disjoint and that $\lambda_j \in [0, \infty)$ for $1 \leq j \leq n$. Then

$$s = \sum_{j=1}^{m} \lambda_j \mathbb{1}_{B_j}$$

is a MNNSF. Furthermore,

$$\mathrm{Int}_E(s) = \sum_{j=1}^{m} \lambda_j \cdot \mu(B_j \cap E).$$

Proof

Proof.

Clearly, s is a MNNSF. We can assume that $\lambda_k \neq 0$ for all k. Let $s(X) \setminus \{0\} = \{\, \alpha_1, \dots, \alpha_n \,\}$ with the α_k distinct. Let $A_k = s^{-1}(\alpha_k)$. Since the B_j are pairwise disjoint, $A_k = \bigcup_{\lambda_j = \alpha_k} B_j$. Then by definition

$$\begin{aligned}
\sum_{j=1}^{m} \lambda_j \cdot \mu(B_j \cap E) &= \sum_{k=1}^{n} \sum_{\lambda_j = \alpha_k} \alpha_k \cdot \mu(B_j \cap E) \\
&= \sum_{k=1}^{n} \alpha_k \cdot \mu(A_k \cap E) \\
&= \mathrm{Int}_E(s).
\end{aligned}$$

□

Linearity of Int_E

Lemma

Let $s, t : X \to [0, \infty)$ be MNNSFs. Then for all $\alpha, \beta \in [0, \infty)$, $\alpha \cdot s + \beta \cdot t$ is a MNNSF, and

$$\text{Int}_E(\alpha \cdot s + \beta \cdot t) = \alpha \,\text{Ind}_E(s) + \beta \cdot \text{Int}(t).$$

Proof.

Let $s = \sum_{k=1}^{n} \alpha_k \mathbb{1}_{A_k}$ and $t = \sum_{j=1}^{m} \beta_j \mathbb{1}_{B_j}$ be the standard forms of s and t, respectively. Let $E_{ij} = A_i \cap B_j$. Then $\{ E_{ij} \}$ is a finite pairwise disjoint family in $\mathcal{M}$.

Proof Continued.

Then $\alpha \cdot s + \beta \cdot t = \sum_{ij}(\alpha\alpha_i + \beta\beta_j) \cdot \mathbb{1}_{E_{ij}}$. Then

$$\begin{aligned}
\operatorname{Int}_E(\alpha \cdot s + \beta \cdot t) &= \sum_{ij}(\alpha\alpha_i + \beta\beta_j) \cdot \mu(E_{ij} \cap E) \\
&= \alpha \sum_{ij} \alpha_i \cdot \mu(E_{ij} \cap E) + \beta \sum_{ij} \beta_j \cdot \mu(E_{ij} \cap E) \\
&= \alpha \sum_{i} \alpha_i \cdot \mu(A_i \cap E) + \beta \sum_{j} \beta_j \cdot \mu(B_j \cap E) \\
&= \alpha \operatorname{Int}_E(s) + \beta \operatorname{Int}_E(t).
\end{aligned}$$

□

Linear Combinations of Characteristic Functions

Corollary

Suppose that $A_1, \dots, A_n \in \mathcal{M}$ *and* $\alpha_1, \dots, \alpha_n \in [0, \infty)$. *Then* $s = \sum_{k=1}^n \alpha_k \cdot \mathbb{1}_{A_k}$ *is a MNNSF, and*

$$\operatorname{Int}_E\Big(\sum_{k=1}^n \alpha_k \cdot \mathbb{1}_{A_k}\Big) = \sum_{k=1}^n \alpha_k \cdot \mu(A_k \cap E).$$

Death to Int_E

Corollary

If s and t are MNNSFs such that $s \leq t$ (that is, $s(x) \leq t(x)$ for all $x \in X$), then

$$\text{Int}_E(s) \leq \text{Int}_E(t).$$

It follows that

$$\text{Int}_E(s) = \int_E s(x)\, d\mu(x).$$

Proof.

We have $t = s + (t - s)$ and $t - s$ is a MNNSF. Thus $\text{Ind}_E(t) = \text{Int}_E(s) + \text{Int}_E(t - s) \geq \text{Int}_E(s)$. Then

$$\int_E s(x)\, d\mu(x) = \sup_{0 \leq t \leq s} \text{Int}_E(t) \leq \text{Int}_E(s).$$

But $s \leq s$. □

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Low Hanging Fruit

Proposition

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f, g : X \to [0, \infty]$ measurable.

1. $f \leq g$ *implies* $\int_E f \, d\mu \leq \int_E g \, d\mu$.
2. *If* $A \subset E$, *then* $\int_A f \, d\mu \leq \int_E f \, d\mu$.
3. *If* $\alpha \in [0, \infty)$, *then* $\alpha \int_E f \, d\mu = \int_E \alpha f \, d\mu$.
4. *If* $f(x) = 0$ *for all* $x \in E$, *then* $\int_E f \, d\mu = 0$ *(even if* $\mu(E) = \infty$*).*
5. *If* $\mu(E) = 0$, *then* $\int_E f \, d\mu = 0$ *(even if* $f(x) = \infty$ *for all* $x \in E$*).*
6. $\int_E f \, d\mu = \int_X \mathbb{1}_E \cdot f \, d\mu$.

Proof.

Let as exercises. □

New Measures from Old

Lemma

Suppose that $(X, \mathcal{M}, \mu)$ *is a measure space and that* $s : X \to [0, \infty)$ *is a MNNSF. Then*

$$\nu(E) = \int_E s(x)\, d\mu(x)$$

defines a measure on $(X, \mathcal{M})$*.*

Proof.

Clearly $\nu(\emptyset) = 0$. Suppose $\{ E_i \} \subset \mathcal{M}$ are pairwise disjoint. Let $s = \sum_k \alpha_k \mathbb{1}_{A_k}$ be the standard representation of s.

Proof

Proof Continued.

Then

$$\begin{aligned}\nu\Big(\bigcup E_i\Big) &= \int_{\bigcup E_i} s\,d\mu = \sum_{k=1}^{n} \alpha_k \cdot \mu\Big(A_k \cap \bigcup E_i\Big) \\ &= \sum_{k=1}^{n} \alpha_k \sum_{i=1}^{\infty} \mu(A_k \cap E_i) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{n} \alpha_k \mu(A_k \cap E_i) \\ &= \sum_{i=1}^{\infty} \int_{E_i} s\,d\mu = \sum_{i=1}^{\infty} \nu(E_i).\end{aligned}$$

□

The Monotone Convergence Theorem

Theorem (Monotone Convergence Theorem)

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $f_n : X \to [0, \infty]$ is measurable for all $n \in \mathbf{N}$. Suppose also that

1. $0 \le f_1 \le f_2 \le \cdots$, *and*
2. $f(x) := \lim_{n\to\infty} f_n(x)$ *for all* $x \in X$.

Then $f : X \to [0, \infty]$ is measurable and

$$\lim_{n\to\infty} \int_E f_n(x)\,d\mu(x) = \int_E f(x)\,d\mu(x) \quad \text{for all } E \in \mathcal{M}. \qquad (\ddagger)$$

Proof.

Replacing f_n by $\mathbb{1}_E \cdot f_n$, we may as well assume $E = X$. Since f is pointwise limit of measurable functions, it too is measurable. Thus we just have to prove $(\ddagger)$.

Proof

Proof.

Since $f_n \leq f_{n+1}$, we have $\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu$. Therefore there is a $\alpha \in [0, \infty]$ such that

$$\lim_n \int_X f_n \, d\mu = \alpha = \sup_{n \geq 1} \int_X f_n \, d\mu.$$

Since each $f_n \leq f$, we have $\int_X f_n \, d\mu \leq \int_X f \, d\mu$ and

$$\alpha \leq \int_X f \, d\mu.$$

Proof Continued.

Let s be a MNNSF such that $0 \leq s \leq f$. Fix $0 < c < 1$ and let $E_n = \{ x \in X : f_n(x) \geq cs(x) \}$. Then $E_n \in \mathcal{M}$ and $E_1 \subset E_2 \subset \cdots$. If $f(x) = 0$, then $s(x) = 0$ and $x \in E_1$. If $f(x) > 0$, then for some n, $f_n(x) > cs(x)$ and $x \in E_n$. Therefore

$$X = \bigcup E_n.$$

Let

$$\nu(E) = \int_E s \, d\mu.$$

Recall that ν is a measure on $(X, \mathcal{M})$.

Proof

Proof Continued.

We have

$$c\int_X s\,d\mu = c\nu(X) = c\lim_n \nu(E_n) = \lim_n \int_{E_n} cs(x)\,d\mu(x)$$
$$\leq \limsup_n \int_{E_n} f_n(x)\,d\mu(x)$$
$$\leq \limsup_n \int_X f_n(x)\,d\mu(x)$$
$$= \lim_n \int_X f_n(x)\,d\mu(x) = \alpha.$$

Therefore $\alpha \geq c\int_X s(x)\,d\mu(x)$ for all $0 \leq s \leq f$. Therefore $\alpha \geq c\int_X f(x)\,d\mu(x)$ for all $0 < c < 1$. Therefore $\alpha \geq \int_X f(x)\,d\mu(x)$. This completes the proof. □

A Concrete Example from the Future

Example

Let $([0,1], \mathcal{B}([0,1]), m)$ be our once and future measure—Lebesgue measure. Let $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \in (0,1], \text{ and} \\ 0 & \text{if } x = 0. \end{cases}$. Let $f_n(x) = \mathbb{1}_{[\frac{1}{n},1]}(x) f(x)$. Then down the road we will see that

$$\int_{[0,1]} f_n(x)\, dm(x) = \mathcal{R}\int_{\frac{1}{n}}^{1} \frac{1}{\sqrt{x}} = 2 - \frac{1}{\sqrt{n}}.$$

Since $f_n \nearrow f$, the MCT implies

$$\int_{[0,1]} \frac{1}{\sqrt{x}}\, dm(x) = 2,$$

and there are no “improper integrals” involved.

Break Time

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- Questions?
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Sums

Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $f_n : X \to [0, \infty]$ is measurable for all $n \in \mathbf{N}$. Then

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

defines a measurable function $f : X \to [0, \infty]$ and

$$\int_X f(x)\, d\mu(x) = \sum_{n=1}^{\infty} \int_X f_n(x)\, d\mu(x).$$

Proof

Proof.

Let (s_n) and (t_n) be MNNSFs such that $s_n \nearrow f_1$ and $t_n \nearrow f_2$. Then by the MCT,

$$\int_X (f_1 + f_2)\, d\mu = \lim_n \int_X (s_n + t_n)\, d\mu = \lim_n \Bigl(\int_X s_n\, d\mu + \int_X t_n\, d\mu \Bigr)$$

$$= \int_X f_1\, d\mu + \int_X f_2\, d\mu.$$

Thus by an induction argument, if $g_N = f_1 + \cdots + f_N$, then

$$\int_X g_N\, d\mu = \sum_{n=1}^{N} \int_X f_n\, d\mu.$$

Proof

Proof Continued.

But $g_N \nearrow f$. Therefore f is measurable and by the MCT again,

$$\begin{aligned}\int_X f(x)\,d\mu(x) &= \lim_{N\to\infty} \int_X g_N(x)\,d\mu(x) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n(x)\,d\mu(x) \\ &= \sum_{n=1}^{\infty} \int_X f_n(x)\,d\mu(x).\end{aligned}$$

□

Corollary

Suppose $f, g : (X, \mathcal{M}) \to [0, \infty]$ *are measurable and that* $\alpha, \beta \in [0, \infty)$. *Then*

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu.$$

That's Enough for Today

- That is enough for now.