# Math 73/103: Fall 2020
# Lecture 14

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October 14, 2020

# Getting Started

- We should be recording!
- As usual, this a good time to ask questions about the previous lecture, complain, or tell a story.
- Our next homework will be due on the 21st. We'll see just what problems when we see where we are Friday.

# The Next Big Theorem

**Theorem (Fatou's Lemma)**

*Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $f_n : X \to [0, \infty]$ is measurable for all $n \in \mathbf{N}$. Then*

$$\int_X (\liminf_n f_n)(x)\, d\mu(x) \le \liminf_n \int_X f_n(x)\, d\mu(x)$$

# Proof

**Proof.**

Let $g_k = \inf_{n\geq k} f_n$. Then $g_k \leq f_k$, and hence $\int_X g_k \, d\mu \leq \int_X f_k \, d\mu$. Therefore

$$\liminf_k \int_X g_k \, d\mu \leq \liminf_k \int_X f_k \, d\mu. \tag{1}$$

But $g_k \nearrow \liminf_k f_k$. Therefore by the MCT

$$\liminf_k \int_X g_k \, d\mu = \lim_k \int_X g_k \, d\mu = \int_X \liminf_k f_k \, d\mu. \tag{2}$$

Combining (1) and (2) completes the proof. □

# Can't Guarantee Equality

## Example

Let $(X, \mathcal{M}, \mu)$ be a measure sapce with $A, B \in \mathcal{M}$ disjoint sets satisfying $\mu(A) > 0$ and $\mu(B) > 0$ .Let

$$f_n(x) = \begin{cases} \mathbb{1}_A(x) & \text{if } n \text{ is even, and} \\ \mathbb{1}_B(x) & \text{if } n \text{ is odd.} \end{cases}$$

Then $\liminf_n f_n = 0$. But

$$\liminf_n \int_X f_n \, d\mu = \min\{\, \mu(A), \mu(B) \,\} > 0.$$

That is, in this case,

$$\int_X \liminf f_n \, d\mu < \liminf_k \int_X f_k \, d\mu.$$

# There is Always Something

**Proposition**

*Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and that $f_n : X \to [0, \infty]$ is a sequence of measurable functions converging pointwise to $f : X \to [0, \infty]$. Then*

$$\int_X f(x)\, d\mu(x) \le \liminf_n \int_X f_n(x)\, d\mu(x).$$

**Proof.**

In this case, $f = \liminf_n f_n$. □

**Remark**

*Using this observation, we see that the MCT follows from the statement of Fatou's Lemma.*

# New Measures from Old (NC-17 Version)

**Theorem**

*Suppose that* $(X, \mathcal{M}, \mu)$ *is a measure space and that* $f : X \to [0, \infty]$ *is measurable. Then*

$$\nu(E) = \int_E f(x)\, d\mu(x) \quad \text{for all } E \in \mathcal{M}$$

*defines a measure on* $(X, \mathcal{M})$*. Moreover*

$$\int_X g(x)\, d\nu(x) = \int_X g(x) f(x)\, d\mu(x)$$

*for all* $g : X \to [0, \infty]$ *measurable.*

**Proof.**

Clearly $\nu(\emptyset) = 0$. So let $\{ E_i \} \subset \mathcal{M}$ be pairwise disjoint. For convenience, let $E = \bigcup E_i$. Then $\mathbb{1}_E \cdot f = \sum_i \mathbb{1}_{E_i} \cdot f$. Therefore

$$\begin{aligned} \nu(E) &= \int_E f \, d\mu = \int_X \mathbb{1}_E \cdot f \, \mu \\ &= \sum_{i=1}^{\infty} \int_X \mathbb{1}_{E_i} \cdot f \, d\mu \\ &= \sum_{i=1}^{\infty} \int_{E_i} f \, d\mu \\ &= \sum_{i=1}^{\infty} \nu(E_i). \end{aligned}$$

Therefore $\nu$ is a measure as claimed.

# Proof

**Proof Continued.**

Now suppose that $g = \mathbb{1}_A$. Then

$$\int_X g\,d\nu = \nu(A) = \int_A f\,d\mu = \int_X g \cdot f\,d\mu.$$

Therefore by linearity

$$\int_X g\,d\nu = \int_X g \cdot f\,d\mu$$

for any MNNSF $g$! But if $g : X \to [0, \infty]$ is arbitrary, then there is a sequence $(g_n)$ of MNNSFs such that $g_n \nearrow g$. Now by the MCT, $\lim_n \int_X g_n\,d\nu = \int_X g\,d\nu$. But we also have $g_n \cdot f \nearrow g \cdot f$. By the MCT again, $\int_X g_n \cdot f\,d\mu \nearrow \int_X g \cdot f\,d\mu$. Since $\int_X g_n\,d\nu = \int_X g_n \cdot f\,d\mu$, this completes the proof. □

# Corollary

## Corollary

*Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and that $A \subset B$ are measurable sets. Then*

$$\int_B f(x)\,d\mu(x) = \int_A f(x)\,d\mu(x) + \int_{B\setminus A} f(x)\,d\mu(x).$$

## Proof.

This follows immediately from the finite additivity of $\nu(E) = \int_E f(x)\,d\mu(x)$. □

# Preview of Coming Attractions

**Remark**

*Notice that if* $\nu = \int f \cdot d\mu$*, then* $\mu(E) = 0$ *implies that* $\nu(E) = 0$*. In this case, we say that* $\nu$ *is absolutely continuous with respect to* $\mu$*. Under modest hypotheses, the converse holds: if* $\nu$ *is absolutely continuous with respect to* $\mu$ *there is a measurable function* $f : X \to [0, \infty]$ *such that*

$$\nu(E) = \int_E f \, d\mu.$$

*The function* $f$ *is called the Radon-Nykodym derivative of* $\nu$ *with respect to* $\mu$*. I hope we will be able to prove this later in the term.*

# Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

**Definition**

Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{L}^1(X, \mathcal{M}, \mu)$ be the collection of measurable functions $f : X \to \mathbf{C}$ such that

$$\int_X |f(x)| \, d\mu(x) < \infty.$$

We call $\mathcal{L}^1(X)$ the Lebesgue integrable functions on $X$.

# Getting Non-Negative

- Recall that if $f : X \to \mathbf{C}$ is measurable, then $f(x) = u(x) + i\,v(x)$ with $u, v : X \to \mathbf{R}$ measurable.
- If $g : X \to [-\infty, \infty]$ is measurable, we let $g^+(x) := \max\{\, g(x), 0 \,\} = \mathbb{1}_{E^+} \cdot g$ where $E^+ = \{\, x \in X : g(x) \geq 0 \,\}$. Similarly, let $g^-(x) = \max\{\, -g(x), 0 \,\}$.
- The point is that $g = g^+ - g^-$ with both $g^\pm : X \to [0, \infty]$ measurable.
- Thus if $f : X \to \mathbf{C}$ is measurable, then $f = u^+ - u^- + i(v^+ - v^-)$ with $u^\pm, v^\pm : X \to [0, \infty)$ measurable.
- Notice that if $f \in \mathcal{L}^1(X)$ and $k \in \{\, u^\pm, v^\pm \,\}$, then $0 \leq k \leq |f|$. This means $\int_X k\, d\mu < \infty$.

# The Integral

## Definition

If $f = u^+ - u^- + i(v^+ - v^-) \in \mathcal{L}^1(X)$ with $u^\pm, v^\pm$ as defined on the previous slide, then we define

$$\int_X f(x)\,d\mu(x) = \int_X u^+(x)\,d\mu(x) - \int_X u^-(x)\,d\mu(x) + i\Big[\int_X v^+(x)\,d\mu(x) - \int_X v^-(x)\,d\mu(x)\Big]$$

return

**Remark**

*Some authors like to consider measurable functions* $g = g^+ - g^- : X \to [-\infty, \infty]$. *Then we can define*

$$\int_X g(x)\,d\mu(x) = \int_X g^+(x)\,d\mu(x) - \int_X g^-(x)\,d\mu(x) \qquad (*)$$

*provided at most one of* $\int_X g^\pm(x)\,d\mu(x)$ *is infinite. We won't bother with this, but even if one does, then saying "*$g$ *is integrable" or "*$g \in \mathcal{L}^1(X)$*" would still entail* $\int_X |g|\,d\mu < \infty$*, and both integrals on the right-hand side of* $(*)$ *would be finite.*

# Vector Space

## Theorem

*If $(X, \mathcal{M}, \mu)$ is a measure space, then $\mathcal{L}^1(X)$ is a complex vector space. Furthermore, if $\alpha, \beta \in \mathbf{C}$ and $f, g \in \mathcal{L}^1(X)$, then*

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu.$$

## Proof.

Since

$$\begin{aligned}\int_X |\alpha f + \beta g|\, d\mu &\le \int_X \big(|\alpha||f| + |\beta||g|\big)\, d\mu \\ &\le |\alpha| \int_X |f|\, d\mu + |\beta| \int_X |g|\, d\mu < \infty,\end{aligned}$$

$\alpha f + \beta g \in \mathcal{L}^1(X)$. Therefore $\mathcal{L}^1(X)$ is a vector space.

## Proof Continued.

Now we want to prove

$$\int_X \alpha f \, d\mu = \alpha \int_X f \, d\mu. \qquad (*)$$

If $\alpha \geq 0$, this is almost immediate from the ▸ definition. If $\alpha = -1$, then

$$\begin{aligned} \int -f &= \int -(u^+ - u^-) - i(v^+ - v^-) \\ &= \int (u^- - u^+) + i(v^- - v^+)) \\ &= \int u^- - \int u^+ + i\Big(\int v^- - \int v^+\Big) = -\int f. \end{aligned}$$

Thus $(*)$ holds for all $\alpha \in \mathbf{R}$. Since a similar computation works with $\alpha = i$, we have established $(*)$.

# Proof

## Proof Continued.

Since we can treat the real and imaginary parts of $f$ separately, it will suffice to see that $\int(f+g) = \int f + \int g$ for $f, g : X \to \mathbf{R}$. So let $h = f + g$. Then $h^+ - h^- = f^+ - f^- + g^+ - g^-$. Thus $h^+ + f^- + g^- = h^- + f^+ + g^+$. Therefore

$$\int h^+ + \int f^- + \int g^- = \int h^- + \int f^+ + \int g^+.$$

Consequently,

$$\int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^{-1}.$$

Therefore $\int h = \int f + \int g$ and we're done. □

## Bounds

**Theorem**

*If $f \in \mathcal{L}^1(X)$, then* $\left| \int_X f(x)\, d\mu(x) \right| \le \int_X |f(x)|\, d\mu(x)$.

**Proof.**

Let $z = re^{i\theta} = \int_X f(x)\, d\mu(x)$. Let $\alpha = e^{-i\theta}$. Then

$$\begin{aligned} \left| \int_X f(x)\, d\mu(x) \right| &= \alpha \int_X f(x)\, d\mu(x) = \int_X \alpha f(x)\, d\mu(x) \\ &= \int_X \operatorname{Re}\bigl(\alpha f(x)\bigr)\, d\mu(x) + \underbrace{i \int_X \operatorname{Im}\bigl(\alpha f(x)\bigr)\, d\mu(x)}_{=0} \end{aligned}$$

# Proof

## Proof Continued.

$$\begin{aligned}\int_X \operatorname{Re}\bigl(\alpha f(x)\bigr)\,d\mu(x) &= \int_X \operatorname{Re}\bigl(\alpha f(x)\bigr)^+\,d\mu(x) \\ &\qquad - \int_X \operatorname{Re}\bigl(\alpha f(x)\bigr)^-\,d\mu(x) \\ &\le \int_X \operatorname{Re}\bigl(\alpha f(x)\bigr)^+\,d\mu(x) \\ &\le \int_X |\alpha f(x)|\,d\mu(x) \\ &= \int_X |f(x)|\,d\mu(x).\end{aligned}$$

□

# Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

# The Dominated Convergence Theorem

## Theorem (Lebesgue's Dominated Convergence Theorem)

*Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $f_n : X \to \mathbf{C}$ is measurable for all $n \in \mathbf{N}$ and that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$. Suppose further that there is a $g \in \mathcal{L}^1(X)$ such that $|f_n(x)| \leq g(x)$ for all $x \in X$. Then $f \in \mathcal{L}^1(X)$ and*

$$\lim_{n\to\infty} \int_X \big|f_n(x) - f(x)\big| \, d\mu(x) = 0. \tag{†}$$

## Remark

*Note that (†) implies that*

$$\lim_{n\to\infty} \int_X f_n(x)\, d\mu(x) = \int_X f(x)\, d\mu(x).$$

*However, (†) is formally a stronger conclusion.*

# Proof

**Proof.**

As the pointwise limit of measurable functions, $f : X \to \mathbf{C}$ is measurable. Since $|f| \le g$, we also easily see that $f \in \mathcal{L}^1(X)$. On the other hand, $|f_n - f| \le 2g$. (Since our functions are $\mathbf{C}$-valued, we don't have to worry about $\infty - \infty$!) Let $g_n = 2g - |f_n - f|$. Then $\liminf_n g_n = \lim_n g_n = 2g$. Since $g_n \ge 0$, we can apply Fatou's Lemma and

$$\begin{aligned}\int_X 2g\,d\mu &\le \liminf_n \int_X \bigl(2g - |f_n - f|\bigr)\,d\mu \\ &= \int_X 2g\,d\mu + \liminf_n \Bigl(-\int_X |f_n - f|\,d\mu\Bigr) \\ &= \int_X 2g\,d\mu - \limsup_n \int_X |f_n - f|\,d\mu\end{aligned}$$

# Proof

**Proof Continued.**

Since $\int_X 2g \, d\mu < \infty$, we conclude that

$$0 \geq \limsup_n \int_X |f_n - f| \, d\mu \geq \liminf_n \int_X |f_n - f| \, d\mu \geq 0.$$

But then it follows that

$$\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0$$

as required. $\square$

# Take That Riemann!

**Example**

Suppose that $f_n : [0,1] \to [0,1]$ are continuous and that $f_n \to 0$ pointwise on $[0,1]$. We can let $g(x) = 1$ for all $x \in [0,1]$. Then by assumption $|f_n(x)| = f_n(x) \le 1 = g(x)$ for all $x \in [0,1]$. Since we will eventually show that the Riemann integral agrees with the Lebesgue integral on $[0,1]$, we have $g \in \mathcal{L}^1([0,1],m)$ and then the LDCT implies that

$$\mathcal{R}\int_0^1 f_n = \int_{[0,1]} f_n \, dm \to \int_{[0,1]} 0 \, dm = 0.$$

# That's Enough for Today

- That is enough for now.