Math 73/103: Fall 2020
Lecture 15

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- We should be recording!
- As usual, this a good time to ask questions about the previous lecture, complain, or tell a story.
- Our next homework will be due on the 21st.
- Legal Cheating on Homework: To construct counterexamples on homework, we can assume that we have defined Lebesgue measure m on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ such that the Lebesgue integral extends the Riemann integral. Thus $\mathbb{1}_{[-n,n]}$ has integral $2n$ and $\mathbb{1}_{[0,\infty)}$ has infinite integral.

Sets of Measure Zero

- If $(X, \mathcal{M}, \mu)$ is a measure space, then sets $N \in \mathcal{M}$ of measure 0 are called null sets.
- If $f, g : X \to \mathbf{C}$ are both measurable, then $N = \{\, x \in X : f(x) \neq g(x) \,\} \in \mathcal{M}$.
- If $\mu(N) = 0$, then we say that $f = g$ almost everywhere and we write $f \sim g$.
- If $f \sim g$ and $g \sim h$, then $N_1 = \{\, x : f(x) \neq g(x) \,\}$ and $N_2 = \{\, x : g(x) \neq h(x) \,\}$ are null sets and $N_3 = \{\, x : f(x) \neq h(x) \,\} \subset N_1 \cup N_2$. Hence $f \sim h$ and $\sim$ is an equivalence relation.
- If $f \sim g$, then

$$\int_E |f - g| \, d\mu = \int_{E \cap N} |f - g| \, d\mu + \int_{E \setminus N} |f - g| \, d\mu = 0 + 0.$$

Thus $\int_E f \, d\mu = \int_E g \, d\mu$!

Disjointification

Lemma

Let $\{ A_n \}_{n=1}^{\infty} \subset \mathcal{P}(X)$*. Let* $B_1 = A_1$ *and define* $B_n = A_n \setminus \bigcup_{k=1}^{n-1} B_k$ *for* $n \geq 2$*. Then the* B_n *are pairwise disjoint and for each* n*,* $B_n \subset A_n$ *and* $\bigcup_{k=1}^{n} A_n = \bigcup_{k=1}^{n} B_k$*. In particular,* $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k$*. If* $(X, \mathcal{M})$ *is a measurable space and each* $A_n \in \mathcal{M}$*, then each* $B_n \in \mathcal{M}$*.*

Proof.

Work this out for homework. □

Proposition

Let $(X, \mathcal{M}, \mu)$ *be a measure space. Then* μ *is countably subadditive; that is, if* $\{ A_n \}_{n=1}^{\infty} \subset \mathcal{M}$*, then*

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Proof.

Use the lemma to find $\{ B_n \}$ as in the lemma. Then

$$\mu\Bigl(\bigcup_n A_n\Bigr) = \mu\Bigl(\bigcup_n B_n\Bigr) = \sum_{n=1}^{\infty} \mu(B_n) \le \sum_{n=1}^{\infty} \mu(A_n). \quad \square$$

Corollary

If $(X, \mathcal{M}, \mu)$ is a measure space, then the countable union of null sets in X is a null set.

Complete Measures

Remark

Later in the term—once we have defined Lebesgue measure $(\mathbf{R}, \mathcal{B}(\mathbf{R}), m)$—we will see that there is a subset $C \subset [0,1]$ that has zero measure with cardinality $|C| = \mathfrak{c} := |\mathbf{R}|$. If you accept that $|\mathcal{B}(\mathbf{R})| = \mathfrak{c}$, then C has subsets that are not Borel. This means that we can have a null set C not all of whose subsets are null sets. Note that any Borel subset of C would be a null set by monotonicty. This is annoying and counter-intuitive.

Definition

A measure space $(X, \mathcal{M}, \mu)$ is said to be complete if $\mu(N) = 0$ implies that every subset of N is measurable (and hence null).

Remark

The point of the above remark is that it will turn out that $(\mathbf{R}, \mathcal{B}(\mathbf{R}), m)$ is not complete.

Completion

Theorem

Let $(X, \mathcal{M}, \mu)$ *be a measure space. Let* $\mathcal{M}^*$ *be the set of subsets* $B \subset X$ *such that there are* $F, G \in \mathcal{M}$ *such that* $F \subset B \subset G$ *and* $\mu(G \setminus F) = 0$*. Then* $\mathcal{M}^*$ *is a* σ*-algebra containing* $\mathcal{M}$ *and there is a complete measure* μ^* *on* $\mathcal{M}^*$ *extending* μ *such that* $\mu^*(B) = \mu(F)$ *whenever* $F \subset B \subset G$ *satisfies* $\mu(G \setminus F) = 0$ *with* $F, G \in \mathcal{M}$*. We call* $(X, \mathcal{M}^*, \mu^*)$ *the completion of* $(X, \mathcal{M}, \mu)$*.*

Proof

Proof.

Clearly, $\mathcal{M} \subset \mathcal{M}^*$, so $X \in \mathcal{M}^*$. If $B \in \mathcal{M}^*$, then let F, G be such that $F \subset B \subset G$ with $\mu(G \setminus F) = 0$. Then $G^C \subset B^C \subset F^C$ and $\mu(F^C \setminus G^C) = \mu(F^C \cap G) = \mu(G \setminus F) = 0$. Therefore $B^C \in \mathcal{M}^*$.

Now suppose $\{ B_n \} \subset \mathcal{M}^*$. Suppose that $F_n \subset B_n \subset G_n$ are such that $\mu(G_n \setminus F_n) = 0$. Then $\bigcup F_n \subset \bigcup B_n \subset \bigcup G_n$ and

$$\mu\Big(\bigcup G_n \setminus \bigcup F_n\Big) = \mu\Big(\bigcup (G_n \setminus \bigcup F_n\Big) \leq \mu\Big(\bigcup (G_n \setminus F_n)\Big) = 0.$$

This shows that $\mathcal{M}^*$ is a σ-algebra containing $\mathcal{M}$.

Proof

Proof Continued.

We need to verify that μ^* is well-defined. Suppose that $F \subset B \subset G$ and $F' \subset B \subset G'$. Let $G'' = G' \cap G$. Then $\mu(G'' \setminus F) = 0 = \mu(G'' \setminus F')$. Suppose that $\mu(G'') = \infty$. Then $\mu(G'') = \mu(F) + \mu(G'' \setminus F) = \mu(F)$ and $\mu(F) = \infty$. By symmetry, $\mu(F') = \infty$ as well. Now suppose that $\mu(G'') < \infty$. Then $\mu(F) = \mu(G'') = \mu(F')$. This shows that μ^* is a well defined set-function on $\mathcal{M}^*$ which extends μ.

Suppose that $\{ B_n \} \subset \mathcal{M}^*$ are pairwise disjoint. Let $F_n \subset B_n \subset G_n$. Note that the F_n are also pairwise disjoint. Then

$$\mu^*\Bigl(\bigcup B_n\Bigr) = \mu\Bigl(\bigcup F_n\Bigr) = \sum_n \mu(F_n) = \sum_n \mu^*(B_n).$$

This completes the proof. □

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

The Importance of Being Complete

Lemma

Suppose that $(X, \mathcal{M}, \mu)$ is a complete measure space and that $g : X \to \mathbf{C}$ is measurable. If $f \sim g$, then f is measurable.

Proof.

Let $N = \{ x : f(x) \neq g(x) \}$. If $V \subset \mathbf{C}$ is open, then

$$f^{-1}(V) = g^{-1}(V) \cap N^C \cup f^{-1}(V) \cap N.$$

Since N is null, so is $f^{-1}(V) \cap N \subset N$. Hence $f^{-1}(V)$ is measurable. □

Limits

Lemma

Suppose that $(X, \mathcal{M}, \mu)$ *is a measure space (which may or may not be complete). Suppose that* $f_n : X \to \mathbf{C}$ *is measurable for all* $n \in \mathbf{N}$*. Then*

$$E := \{\, x \in X : \lim_{n\to\infty} f_n(x) \text{ exists} \,\}$$

is measurable.

Proof.

Since we can consider the real and imaginary parts of f separately, we may as well assume that f is real-valued. Then $g = \limsup_n f_n$ and $h = \liminf_n f_n$ are measurable from X to $[-\infty, \infty]$. But then

$$E = \{\, x : g(x) = h(x) \,\} \setminus \Big(\{\, x : g(x) = \infty \,\} \cup \{\, x : h(x) = -\infty \,\} \Big).$$

Thus suffices (with a little help from HW#31. □

Almost Everywhere Convergence

Corollary

Suppose that $(X, \mathcal{M}, \mu)$ is a complete measure space and that $f_n : X \to \mathbf{C}$ is measurable for all $n \in \mathbf{N}$. Suppose that $f : X \to \mathbf{C}$ is such that there is a null set N such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \notin N$. (We say that (f_n) converges to f "pointwise almost everywhere" or "for almost all x".) Then f is measurable.

Proof.

Let $E = \{\, x : \lim_n f_n(x) \text{ exists} \,\}$. Now $\mathbb{1}_E \cdot f_n \to \mathbb{1}_E \cdot f$. Then $\mathbb{1}_E \cdot f$ is measurable and $\mathbb{1}_E \cdot f \sim f$. □

LDCT Revisited

Theorem (LDCT Revisted)

Let $(X, \mathcal{M}, \mu)$ be a measure space and assume that (f_n) is a sequence of complex-valued measurable functions on X converging almost everywhere to a function $f : X \to \mathbf{C}$. (If μ is not complete, then we must assume that f is measurable—otherwise this is automatic.) Suppose that there is a $g \in \mathcal{L}^1(X)$ such that for each $n \in \mathbf{N}$, $|f_n(x)| \leq g(x)$ for almost all x. Then

$$\lim_n \int_X |f_n - f| \, d\mu = 0.$$

Proof

Proof.

Let N_0 be a null set such that $f_n(x) \to f(x)$ if $x \notin N_0$. Let N_n be a null set such that $|f_n(x)| \leq g(x)$ if $x \notin N_n$. Let $N = \bigcup_{n=0}^{\infty} N_n$. Then N is a null set. Let $E = N^C = X \setminus N$. (We say that E is conull.) Then $\mathbb{1}_E \cdot f_n \to \mathbb{1}_E \cdot f$ pointwise and the $\mathbb{1}_E \cdot f_n$ are dominated by g. Since $|\mathbb{1}_E \cdot f_n - \mathbb{1}_E \cdot f| \sim |f_n - f|$, our previous version of the LDCT implies that

$$\int_X |f_n - f| \, d\mu = \int_X |\mathbb{1}_E \cdot f_n - \mathbb{1}_E \cdot f| \, d\mu \to 0.$$

□

Remark

Clearly, the Monotone Convergence Theorem and Fatou's Lemma can also be spiced up with suitably sprinkled "almost everywhere"'s as needed.

Proposition

Suppose that $(X, \mathcal{M}, \mu)$ is a measure space.

1. *If $f : X \to [0, \infty]$ is measurable and*

$$\int_E f(x)\, d\mu(x) = 0,$$

   then $f(x) = 0$ for almost all $x \in E$.

2. *If $f : X \to \mathbf{C}$ is measurable and*

$$\int_E f(x)\, d\mu(x) = 0$$

   for all $E \in \mathcal{M}$ then $f \sim 0$; that is, $f(x) = 0$ for almost all $x \in X$.

Proof

Proof.

(1) Let $A_n = \{\, x \in E : f(x) \geq \frac{1}{n} \,\}$. Then

$$\frac{1}{n}\mu(A_n) \leq \int_{A_n} f(x)\,d\mu(x) \leq \int_E f(x)\,d\mu(x) = 0.$$

Therefore $\mu(A_n) = 0$ for all $n \in \mathbf{N}$. But $A_n \subset A_{n+1}$ and

$$\mu(\{\, x \in E : f(x) > 0 \,\}) = \mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \lim_n \mu(A_n) = 0.$$

Proof

Proof Continued.

(2) Let $E = \{ x : \mathrm{Re}(f(x)) > 0 \}$. By assumption,

$$\int_E f(x)\,d\mu(x) = 0.$$

Hence

$$\begin{aligned} 0 &= \int_E \mathrm{Re}(f(x))\,d\mu(x) \\ &= \int_E \mathrm{Re}(f(x))^+\,d\mu(x) + \int_{E^C} \mathrm{Re}(f(x))^+\,d\mu(x) \\ &= \int_X \mathrm{Re}(f(x))^+\,d\mu(x). \end{aligned}$$

By part (1), $\mathrm{Re}(f(x))^+ \sim 0$. Similarly, $\mathrm{Re}(f(x))^- \sim 0$ as well as $\mathrm{Im}(f(x))^\pm \sim 0$. Hence $f \sim 0$ as claimed. □

Break Time

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Time to Get a Real Measure

- In order to get a non-trivial measure—in our case, Lebesgue measure—we need to start with a weak substitute.

Definition

If X is a set, then an outer measure on X is a set function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

1. (non-trivial) $\mu^*(\emptyset) = 0$,
2. (monotonic) $A \subset B$ implies $\mu^*(A) \leq \mu^*(B)$, and
3. (countably subadditive) $\mu^*\big(\bigcup_{n=1}^{\infty} A_n\big) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

▸ return

Remark

If μ^* is an outer measure, then property (1) implies μ^* is finitely subadditive. Using "disjointification", it folows that a set function satisfying (1), (2), and (3) for disjoint unions is an outer measure.

Lebesgue Outer Measure

Definition

Define $m^* : \mathcal{P}(\mathbf{R}) \to [0, \infty]$ by

$$m^*(A) = \inf\Big\{ \sum_{n=1}^{\infty} \ell(I_n) : A \subset \bigcup_{n=1}^{\infty} I_n \quad \text{where each } I_n \text{ is an open interval} \Big\}.$$

Remark

Here we allow unbounded open intervals (a, ∞), $(-\infty, a)$ *and even* $\mathbf{R} = (-\infty, \infty)$. *Of course, if* I *is unbounded, then* $\ell(I) = \infty$. *We also allow* $I = \emptyset$ *with* $\ell(I) = 0$ *in that case. Similarly, we can make sense out of* $\ell(I)$ *for any interval, open or not, in* $\mathbf{R}$.

Lebesgue Outer Measure is an Outer Measure

Lemma

The function $m^* : \mathcal{P}(\mathbf{R}) \to [0, \infty]$ defined on the previous slide is an outer measure on $\mathbf{R}$ which we call Lebesgue outer measure on $\mathbf{R}$.

Proof.

Clearly, $m^*(\emptyset) = 0$. Let $\{ A_n \}_{n=1}^{\infty} \subset \mathcal{P}(\mathbf{R})$. If $m^*(A_n) = \infty$ for some n, then (3) in the definition is clear. So we can assume $m^*(A_n) < \infty$ for each n and find open intervals $\{ I_{n,k} \}$ such that $A_n \subset \bigcup_k I_{n,k}$ and $\sum_k \ell(I_{n,k}) < m^*(A_n) + \frac{\epsilon}{2^n}$. Then $\bigcup A_n \subset \bigcup_n \bigcup_k I_{n,k}$ and

$$m^*\Bigl(\bigcup A_n\Bigr) \leq \sum_n \sum_k \ell(I_{n,k}) = \sum_n m^*(A_n) + \frac{\epsilon}{2^n}$$

$$= \sum_{n=1}^{\infty} m^*(A_n) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we're done. □

Getting There

Proposition

Let I be an interval in $\mathbf{R}$. Then $m^*(I) = \ell(I)$.

Sketch of the Proof.

Suppose $I = [a, b]$. If $\epsilon > 0$, then $[a, b] \subset (a - \epsilon, b + \epsilon)$, so $m^*([a, b]) \leq b - a + 2\epsilon$. Since $\epsilon > 0$ is arbitray, $m^*([a, b]) \leq b - a$. To get the reverse inequality, suppose $\{ I_k \}$ is a cover of $[a, b]$ by open intervals. Since I is compact, there is a $n \in \mathbf{N}$ such that $[a, b] \subset \bigcup_{k=1}^n I_k$. It will suffice to prove that $\sum_{k=1}^n \ell(I_k) \geq b - a$. Since a must be in some I_k, there is a $(a_1, b_1) \in \{ I_k \}$ such that $a_1 < a < b_1$. If $b_1 > b$, then $b - a \leq \ell((a_1, b_1)) \leq \sum_k \ell(I_k)$ and we're done. Otherwise there is a $(a_2, b_2) \in \{ I_k \}$ such that $a_2 < b_1 < b_2$. We can continue in this way until we get $a_N < b_{N-1} < b_N$ with $b < b_N$ and $N \leq n$. Then $\sum_{k=1}^n \ell(I_k) \geq \sum_{k=1}^N \ell((a_k, b_k)) = (b_N - a_N) + \cdots + (b_1 - a_1) = b_N - (a_N - b_{N-1}) - \cdots - (a_2 - b_1) - a_1 \geq b_N - a_1 \geq b - a$.

Proof

Proof Continued.

Now if I is unbounded, then it contains closed intervals of arbitrary large length. My monotonicity, $m^*(I) = \infty = \ell(I)$ in this case.

Now if I is any bounded interval, then given $\epsilon > 0$ we can find closed intervals J_1 and J_2 such that $J_1 \subset I \subset J_1$ with $\ell(I) - \epsilon < \ell(J_1)$ and $\ell(I) = \ell(J_2)$. Then by monotonicity,

$$\ell(I) - \epsilon < \ell(J_1) = m^*(J_1) \leq m^*(I) \leq m^*(J_2) = \ell(J_2) = \ell(I).$$

Since $\epsilon > 0$ is arbitrary, it follows that $m^*(I) = \ell(I)$. □

That's Enough for Today

- That is enough for now.