Math 73/103: Fall 2020
Lecture 16

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Getting Started

- We should be recording!
- Questions?
- Our next homework (problems 24-35) will be due Wednesday via `gradescope`.
- Legal Cheating on Homework: To construct counterexamples on homework, we can assume that we have defined Lebesgue measure m on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ such that the Lebesgue integral extends the Riemann integral. Thus $\mathbb{1}_{[-n,n]}$ has integral $2n$ and $\mathbb{1}_{[0,\infty)}$ has infinite integral.
- I probably won't look at 33.2 at all. You need to work with integrals not-necessarily integrable real-valued functions and I promised you wouldn't have to do that. My bad.
- Do we really need a midterm?
- I added a discussion page in Canvas. Mostly for homework, but could be for anything.

μ^*-Measurable Sets

Definition

Suppose that μ^* is an outer measure on a set X. Then we say $E \subset X$ is μ^*-measurable if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C) \quad \text{for all } A \subset X.$$

We write $\mathcal{M}^*$ for the collection of all μ^*-measurable subsets of X.

Remark

By (finite) subadditivity, we always have $\mu^*(A) \le \mu^*(A \cap E) + \mu^*(A \cap E^C)$. So to verify that $E \in \mathcal{M}^*$ we just need to see that

$$\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^C) \quad \textit{for all } A \subset X. \qquad (\dagger)$$

Furthermore, we only have to consider $(\dagger)$ for A with $\mu^*(A) < \infty$.

Getting a Measure

Theorem

Suppose that μ^* is an outer measure on a set X and that $\mathcal{M}^*$ is the collection of μ^*-measurable subsets. Then $\mathcal{M}^*$ is a σ-algebra and $\mu = \mu^*|_{\mathcal{M}^*}$ is a complete measure on $(X, \mathcal{M}^*)$.

Proof.

Clearly $X \in \mathcal{M}^*$, and if $E \in \mathcal{M}^*$, then so is E^C. So it suffices to check countable subadditivity.

Let $E_1, E_2 \in \mathcal{M}^*$ and $A \subset X$. then

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^C)$$
$$\mu^*(A \cap E_1^C) = \mu^*(A \cap E_1^C \cap E_2) + \mu^*(A \cap \underbrace{E_1^C \cap E_2^C}_{=(E_1 \cup E_2)^C})$$

return

Proof

Proof Continued.

Note that
$A\cap(E_1\cup E_2) = (A\cap E_1)\cup[A\cap(E_2\setminus E_1)] = (A\cap E_1)\cup[A\cap(E_2\cap E_1^C)]$.
Thus,

$$\mu^*(A\cap(E_1\cup E_2)) \le \mu^*(A\cap E_1) + \mu^*(A\cap E_1^C\cap E_2).$$

Then, using the equalities on the previous slide,

$$\begin{aligned}\mu^*(A) &= \mu^*(A\cap E_1) + \mu^*(A\cap E_1^C\cap E_2) + \mu^*(A\cap(E_1\cup E_2)^C) \\ &\ge \mu^*(A\cap(E_1\cup E_2)) + \mu^*(A\cap(E_1\cup E_2)^C).\end{aligned}$$

Since A was arbitrary, we have $E_1\cup E_2\in\mathcal{M}^*$. It follows that $\mathcal{M}^*$ is an algebra—that is, $\mathcal{M}^*$ satisfies the axioms of a σ-algebra except that it is only closed under finite unions. (Therefore it is also closed under finite intersections and set difference.)

Proof Continued.

Now assume $E = \bigcup_{n=1}^{\infty} E_n$ with $E_n \in \mathcal{M}^*$. Since $\mathcal{M}^*$ is an algebra, $B_n = E_n \setminus \bigcup_{k=1}^{n-1} E_k \in \mathcal{M}^*$. Thus we can "disjointify" and assume from the onset that $E_n \cap E_m = \emptyset$ if $n \neq m$.

Let $G_n = \bigcup_{k=1}^{n} E_k$. Then $G_n \in \mathcal{M}^*$. Thus if $A \subset X$,

$$\begin{aligned} \mu^*(A) &= \mu^*(A \cap G_n) + \mu^*(A \cap G_n^C) \\ &\geq \mu^*(A \cap G_n) + \mu^*(A \cap E^C). \qquad (\dagger) \end{aligned}$$

Since $E_n \in \mathcal{M}^*$,

$$\begin{aligned} \mu^*(A \cap G_n) &= \mu^*(A \cap G_n \cap E_n) + \mu^*(A \cap G_n \cap E_n^C) \\ &= \mu^*(A \cap E_n) + \mu^*(A \cap G_{n-1}). \end{aligned}$$

By induction,

$$\mu^*(A \cap G_n) = \sum_{k=1}^{n} \mu^*(A \cap E_k). \qquad (\ddagger)$$

Combining $(\dagger)$ with $(\ddagger)$ gives

Proof

Proof Continued.

$$\mu^*(A) \geq \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap E^C).$$

Since this holds for all n,

$$\begin{aligned}\mu^*(A) &\geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \cap E^C) \\ &\geq \mu^*\Big(\bigcup_{k=1}^{\infty} A \cap E_k\Big) + \mu^*(A \cap E^C) \\ &= \mu^*(A \cap E) + \mu^*(A \cap E^C).\end{aligned}$$

This shows that $\mathcal{M}^*$ is a σ-algebra. All that remains is to show that $\mu = \mu^*|_{\mathcal{M}^*}$ is a measure. Since $\mu(\emptyset) = \mu^*(\emptyset) = 0$, we need to see that μ is countably additive.

Proof

Proof Continued.

Suppose that $\{ E_k \} \subset \mathcal{M}^*$ are pairwise disjoint. Then

$$\begin{aligned} \mu(E_1 \cup E_2) &= \mu^*(E_1 \cup E_2) \\ &= \mu^*((E_1 \cup E_2) \cap E_1) + \mu^*((E_1 \cup E_2) \cap E_1^C) \\ &= \mu^*(E_1) + \mu^*(E_2) \\ &= \mu(E_1) + \mu(E_2). \end{aligned}$$

Therefore μ is finitely additive.

Proof

Proof Continued.

Let $E = \bigcup_{k=1}^{\infty} E_k$. On the one hand,

$$\mu(E) = \mu^*\Bigl(\bigcup E_k\Bigr) \leq \sum_k \mu^*(E_k) = \sum_k \mu(E_k).$$

On the other hand, for every n,

$$\mu(E) \geq \mu\Bigl(\bigcup_{k=1}^{n} E_k\Bigr) = \sum_{k=1}^{n} \mu(E_k).$$

Thus

$$\mu(E) \geq \sum_{k=1}^{\infty} \mu(E_k).$$

Therefore we have shown that μ is a measure.

Completeness

Proof Continued.

To see that μ is complete, it suffice to see that $\mu^*(E) = 0$ implies that $E \in \mathcal{M}^*$. But if $A \subset X$, then

$$\begin{aligned} \mu^*(A) &= \mu^*(E) + \mu^*(A) \\ &\geq \mu^*(A \cap E) + \mu^*(A \cap E^C). \end{aligned}$$

Since A was arbitrary, this shows $E \in \mathcal{M}^*$ and we're done. □

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Definition

Let m^* be Lebesgue outer measure on $\mathbf{R}$ and let $\mathcal{L}$ be the σ-algebra of m^*-measurable sets in $\mathbf{R}$. We call $\mathcal{L}$ the Lebesgue measurable sets and $m = m^*|_{\mathcal{L}}$ Lebesgue measure on $(\mathbf{R}, \mathcal{L})$.

Lemma

For all $a \in \mathbf{R}$, $(a, \infty) \in \mathcal{L}$.

Proof.

Fix $a \in \mathbf{R}$. Since points have zero outer measure, note that for any $A' \subset \mathbf{R}$ and $A := A' \setminus \{a\}$,

$$m^*(A) \leq m^*(A') \leq m^*(A) + m^*(\{a\}) = m^*(A).$$

Hence $m^*(A) = m^*(A')$.

Proof

Proof Continued.

To see that (a, ∞) is m^*-measurable, we need to see that for any $A \subset \mathbf{R}$,

$$m^*(A) \geq m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a]))$$

Using the observation on the previous slide, we can replace A by $A \setminus \{a\}$ and assume $a \notin A$. Then we need to show that

$$m^*(A) \geq m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a)))$$

Thus is $\{ I_k \}$ is a collection of open intervals covering A, it will suffice to see that

$$\sum_k \ell(I_k) \geq m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a))).$$

Proof

Proof Continued.

Let $I_k' = I_k \cap (a, \infty)$ and $I_k'' = I_k \cap (-\infty, a)$. Then $\ell(I_k) = \ell(I_k') + \ell(I_k'')$. Furthermore,

$$m^*(A \cap (a, \infty)) \leq \sum_k \ell(I_k') \quad \text{and} \quad m^*(A \cap (-\infty, a)) \leq \sum_k \ell(I_k'').$$

Therefore

$$\begin{aligned} m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a)) &\leq \sum_k \ell(I_k') + \sum_k \ell(I_k'') \\ &= \sum_k \ell(I_k). \end{aligned}$$

This completes the proof. □

Ta Da

Proposition

Every Borel subset of $\mathbf{R}$ *is Lebesgue measurable. In particular, every interval* $I \subset \mathbf{R}$ *is Lebesgue measurable and* $m(I) = \ell(I)$.

Proof.

Since $\mathcal{L}$ is a σ-algebra containing (a, ∞) for all a, we also have $(-\infty, a] \in \mathcal{L}$ for all a. Then so is

$$(-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - \tfrac{1}{n}].$$

Then $(a, b) = (-\infty, b) \cap (a, \infty) \in \mathcal{L}$ for $a < b$. Since every open set is a countable union of intervals, every open set is in $\mathcal{L}$. Since $\mathcal{L}$ is a σ-algebra, $\mathcal{B}(\mathbf{R}) \subset \mathcal{L}$. Since every interval is Borel (why?) and $m^*(I) = \ell(I) = m(I)$ for every interval, we're done. □

Translation Invariant

Proposition

Let m be Lebesgue measure on $(\mathbf{R}, \mathcal{L})$. Then m is translation invariant. That is if $E \in \mathcal{L}$ and $E + y = \{ x + y : x \in E \}$, then $E + y \in \mathcal{L}$ and $m(E + y) = m(E)$.

Proof.

Since it is clear that the $m^*(E) = m^*(E + y)$ for any $E \subset \mathbf{R}$, it suffices to see that $E + y \in \mathcal{L}$ if $E \in \mathcal{L}$. If $A \subset \mathbf{R}$, then since $E \in \mathcal{L}$

$$\begin{aligned} m^*(A) &= m^*(A - y) \\ &= m^*((A - y) \cap E) + m^*(A - y) \cap E^C) \\ &= m^*(A \cap (E + y)) + m^*(A \cap (E^C + y)) \\ &= m^*(A \cap (E + y)) + m^*(A \cap (E + y)^C). \end{aligned}$$

Since A was arbitrary, $E + y \in \mathcal{L}$. □

So What Do We Know?

Remark

We know that $(\mathbf{R}, \mathcal{L}, m)$ *is a complete measure space with* $\mathcal{B}(\mathbf{R}) \subset \mathcal{L}$*. So if we accept that* $(\mathbf{R}, \mathcal{B}(\mathbf{R}), m)$ *(really* $m|_{\mathcal{B}(\mathbf{R})}$*) can't be complete for cardinality reasons, then we have*

$$\mathcal{B}(\mathbf{R}) \subsetneq \mathcal{L}.$$

At the moment, it is possible that $\mathcal{L} = \mathcal{P}(\mathbf{R})$*. Very shortly we will see that—assuming the axiom of choice—*$\mathcal{L} \subsetneq \mathcal{P}(\mathbf{R})$*. But we are getting ahead of ourselves.*

Restriction

Remark

If $(X, \mathcal{M}, \mu)$ *is a measure space and if* $E \in \mathcal{M}$*, then* $\mathcal{M}(E) = \{ A \cap E : A \in \mathcal{M} \}$ *is a* σ*-algebra in* E *which we can also view as a subset of* $\mathcal{M}$*. In particular,* $\mu' = \mu|_{\mathcal{M}(E)}$ *is a measure on* $(E, \mathcal{M}(E))$*. Therefore if* $E \in \mathcal{L}$*, then we get, by restriction, a measure on* $(E, \mathcal{L}(E))$ *which is also called Lebesgue measure. I will usually just write* m *for this measure as well. For example, we can speak of Lebesgue measure on* $\big([a, b], \mathcal{L}([a, b])\big)$*.*

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

A Non-Measurable Set

Let $X = [0,1)$ and define $\oplus : X \times X \to X$ by

$$x \oplus y = \begin{cases} x + y & \text{if } x + y < 1, \text{ and} \\ x + y - 1 & \text{if } x + y \geq 1. \end{cases}$$

Remark

This is all a bit easier to visualize if we identify $X = [0,1)$ with the circle $x^2 + y^2 = 1$ in the plane.

Translation Invariance Again

Lemma

If $E \subset [0,1)$ is (Lebesgue) measurable, then so is $E \oplus y = \{ x \oplus y : x \in E \}$ for any $y \in [0,1)$. Furthermore, $m(E \oplus y) = m(E)$.

Proof.

Let $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1)$. Then $m(E) = m(E_1) + m(E_2)$. But $E_1 \oplus y = E_1 + y$ and $E_2 \oplus y = E_2 + y - 1$. Therefore $E \oplus y \in \mathcal{L}$ and $(E_1 \oplus y) \cap (E_2 \oplus y) = \emptyset$. Thus

$$\begin{aligned} m(E \oplus y) &= m(E_1 \oplus y) + m(E_2 \oplus y) \\ &= m(E_1) + m(E_2) \\ &= m(E). \end{aligned}$$

□

Axiom of Choice

- Define an equivalence relation on $[0,1)$ by $x \sim y$ if $x - y \in \mathbf{Q}$.
- Using the axiom of choice, we can form a set $P \subset [0,1)$ such that P contains exactly one member of each equivalence class in $[0,1)$.
- Let $\{ r_k \}_{k=0}^{\infty}$ be an enumeration of the rationals in $[0,1)$ with $r_0 = 0$.
- Let $P_k = P \oplus r_k$.

The P's Have it.

Lemma

The $\{ P_k \}_{k=0}^{\infty}$ *form a countable partition of* $[0, 1)$.

Proof.

Suppose $x \in P_i \cap P_j$. Ten $x = p_i \oplus r_i = p_j \oplus r_j$. Then $p_i - p_j \in \mathbf{Q}$. This means that $p_i \sim p_j$ and hence that $i = j$. Therefore the P_k are pairwise disjoint. However if $x \in [0, 1)$, then x belongs to some equivalence class. Thus there is a $p \in P$ such that $x - p = r \in \mathbf{Q}$. If $r \geq 0$, then $r = r_k \in [0, 1) \cap \mathbf{Q}$ and $x = p \oplus r_k \in P_k$ for some k. If $r < 0$, then $1 + r = r_k \in [0, 1) \cap \mathbf{Q}$ and $p \oplus r_k = p + r_k - 1 = p + r = x$ and $x \in P_k$. Therefore $\bigcup_{k=0}^{\infty} P_k = [0, 1)$. □

The Punchline

Theorem

The set $P \subset [0,1)$ constructed on the previous slide is not in $\mathcal{L}$. Therefore the Lebesgue measurable sets are a proper subset of $\mathcal{P}(\mathbf{R})$.

Proof.

Suppose to the contrary that $P \in \mathcal{L}$. Then $P_k \in \mathcal{L}$ for all $k \geq 0$ and $m(P_k) = m(P)$ for all k. Then

$$1 = m([0,1)) = m\Big(\bigcup P_k\Big) = \sum_{k=0}^{\infty} m(P_k) = \sum_{k=0}^{\infty} m(P).$$

This leads to a contradiction. Hence $P \notin \mathcal{L}$. □

The Darn Things are Everywhere

Lemma

Let $P \subset [0,1)$ be our non-measurable set from the previous slide. If $E \subset P$ is measurable, them $m(E) = 0$.

Proof.

Let $E_k = E \oplus r_k \subset P \oplus r_k = P_k$. Then
$1 = m([0,1)) \geq m\Big(\bigcup E_k\Big) = \sum_{k=0}^{\infty} m(E)$. Hence $m(E) = 0$. □

Lemma

Suppose that $m^*(A) > 0$. Then A contains a nonmeasurable set.

Proof

Proof.

Suppose $A \subset [0,1)$. Let $A_k = A \cap P_k$. if $A_k \in \mathcal{L}$ for all k, then $m(A_k) = 0$ and

$$0 = \sum_{k=0}^{\infty} m(A_k) = m\Big(\bigcup A_k\Big) = m(A) = m^*(A) > 0.$$

This is a contradiction, so the result holds in this case.

In general, let $E_n = A \cap [n, n+1)$. Then for some n, $m^*(E_n) > 0$. Let $A' = E_n - n \subset [0,1)$. Then by the first part of the proof, there is a nonmeasurable subset $B \subset A'$. But then $B + n \subset E_n \subset A$ is also not measurable. □

That's Enough for Today

- That is enough for now.