Math 73/103: Fall 2020
Lecture 17

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Getting Started

- We should be recording!
- Questions?
- Our next homework is due today.

Promises Kept

Theorem

Suppose that f is a bounded real-valued function on $[a, b]$.

1. *If f is Riemann integrable, then f is Lebesgue measurable and $f \in \mathcal{L}^1([a, b])$. Furthermore,*

$$\mathcal{R}\int_a^b f = \int_{[a,b]} f(x)\, dm(x). \qquad (*)$$

2. *Furthermore, f is Riemann integrable if and only if the set of discontinuities of f has measure zero.*

Remark

In view of this result, we will retire the notation $\mathcal{R}\int_a^b f$ and simply write $\int_a^b f(x)\, dx$ in place of either side of $(*)$.

Proof.

Let f be a bounded real-valued function on $[a, b]$. If $\mathcal{P} = \{ a = t_0 < t_1 < \cdots < t_n = b \}$ is any partition of $[a, b]$ let

$$l_{\mathcal{P}} := \sum_{k=1}^{n} m_k \mathbb{1}_{(t_{k-1},t_k]} \quad \text{and} \quad u_{\mathcal{P}} := \sum_{k=1}^{n} M_k \mathbb{1}_{(t_{k-1},t_k]}$$

where $m_k = \inf_{x\in[t_{k-1},t_k]} f(x)$ and $M_k = \sup_{x\in[t_{k-1},t_k]} f(x)$. Hence

$$\int_{[a,b]} l_{\mathcal{P}}\, dm = \mathcal{L}(f,\mathcal{P}) \quad \text{and} \quad \int_{[a,b]} u_{\mathcal{P}}\, dm = \mathcal{U}(f,\mathcal{P}).$$

Since f is bounded, we can find partitions $\mathcal{Q}_k$ and $\mathcal{R}_k$ such that

$$\lim_k \mathcal{L}(f,\mathcal{Q}_k) = \underline{\mathcal{R}}\int_a^b f \quad \text{and} \quad \lim_k \mathcal{U}(f,\mathcal{R}_k) = \overline{\mathcal{R}}\int_a^b f. \qquad (\ddagger)$$

return

Proof Continued.

Let $\mathcal{P}_k$ be a refinement of $\mathcal{Q}_k$, $\mathcal{R}_k$, and $\mathcal{P}_{k-1}$ such that $\|\mathcal{P}_k\| < \frac{1}{k}$. Then (1) on the previous slide holds with $\mathcal{Q}_k$ and $\mathcal{R}_k$ replaced by $\mathcal{P}_k$. Since $\mathcal{P}_{k+1}$ is a refinement of $\mathcal{P}_k$, we also have

$$l_{\mathcal{P}_k} \leq l_{\mathcal{P}_{k+1}} \leq f \leq u_{\mathcal{P}_{k+1}} \leq u_{\mathcal{P}_k}.$$

Since f is bounded, we get bounded measurable functions $u = \inf_k u_{\mathcal{P}_k} = \lim_k u_{\mathcal{P}_k}$ and $l = \sup_k l_{\mathcal{P}_k} = \lim_k l_{\mathcal{P}_k}$ such that $l \leq f \leq u$. Since bounded measurable functions are integrable on $[a, b]$, the LDCT implies that

$$\int_{[a,b]} l \, dm = \lim_k \int_{[a,b]} l_{\mathcal{P}_k} \, dm = \lim_k \mathcal{L}(f, \mathcal{P}_k) = \underline{\mathcal{R}}\int_a^b f.$$

Similarly,

$$\int_{[a,b]} u \, dm = \overline{\mathcal{R}}\int_a^b f.$$

Proof of (1)

Proof Continued.

(1) Now assume $f \in \mathcal{R}[a,b]$. Then

$$\int_{[a,b]} l\,dm = \underline{\mathcal{R}}\int_a^b f = \overline{\mathcal{R}}\int_a^b f = \int_{[a,b]} u\,dm. \tag{1}$$

Since $l \leq f \leq u$, this implies $u - l \geq 0$ and

$$\int_{[a,b]} (u-l)\,dm = 0$$

Therefore $u - l$ is zero almost everywhere and $l = f = u$ almost everywhere. Since Lebesgue measure is complete, f is measurable and

$$\mathcal{R}\int_a^b f = \int_{[a,b]} l\,dm = \int_{[a,b]} f\,dm.$$

This proves part (1). □

Before proving part (2) of the theorem, it will be useful to make some observations. Note that as a function of $\delta > 0$, $F(\delta) = \sup\{\, f(y) : |y - x| \leq \delta \,\}$ is decreasing: that is, $0 < \delta' < \delta$ implies $f(x) \leq F(\delta') \leq F(\delta)$. Then we can define a function H on $[a, b]$ by

$$H(x) = \lim_{\delta \searrow 0} \sup\{\, f(y) : |y - x| \leq \delta \,\} = \inf_{\delta > 0} F(\delta).$$

Similarly we get h defined on $[a, b]$ by

$$h(x) = \lim_{\delta \searrow 0} \inf\{\, f(y) : |y - x| \leq \delta \,\}.$$

Notice that

$$h(x) \leq f(x) \leq H(x) \quad \text{for all } x \in [a, b].$$

Lemma 1

Lemma

f is continuous at $x \in [a, b]$ if and only if $h(x) = H(x)$.

Proof.

Suppose that f is continuous at x. Then given $\epsilon > 0$, there is a $\delta > 0$ such that $|y - x| < \delta$ implies $f(x) - \epsilon < f(y) < f(x) + \epsilon$. Then $H(x) \leq f(x) + \epsilon$ and $f(x) - \epsilon \leq h(x)$. Since $\epsilon > 0$ is arbitrary, $f(x) \leq h(x) \leq f(x) \leq H(x) \leq f(x)$, and $h(x) = H(x)$.

Conversely, if $h(x) = H(x)$, then they both must equal $f(x)$. Then given $\epsilon > 0$ there is a $\delta > 0$ such that

$$f(x) + \epsilon = H(x) + \epsilon > \sup\{\, f(y) : |y - x| \leq \delta \,\} \quad \text{and}$$
$$f(x) - \epsilon = h(x) - \epsilon < \inf\{\, f(y) : |y - x| \leq \delta \,\}.$$

Therefore $|y - x| < \delta$ implies $f(x) - \epsilon < f(y) < f(x) + \epsilon$. That is, $|f(y) - f(x)| < \epsilon$. □

Lemma 2

Lemma

Both H and h are Lebesgue measurable and

$$\int_{[a,b]} H\,dm = \overline{\mathcal{R}}\int_a^b f \quad \textit{and} \quad \int_{[a,b]} h\,dm = \underline{\mathcal{R}}\int_a^b f.$$

Proof.

Let $\{\mathcal{P}_k\}$ be the nested partitions from the first part of the proof. Let $N = \bigcup \mathcal{P}_k$. Then N is countable and has Lebesgue measure zero. Fix k. If $x \notin N$, then $x \notin \mathcal{P}_k = \{t_0 = a < \cdots < t_n = b\}$ and there is a $\delta > 0$ such that $\{y : |y - x| \leq \delta\} \subset (t_{i-1}, t_i)$. Then

$$M_i := \sup\{f(y) : y \in [t_{i-1}, t_i]\} \geq \sup\{f(y) : |y - x| \leq \delta\} \geq H(x)$$

That is, $u_{\mathcal{P}_k}(x) \geq H(x)$. Since k was arbitrary, $u(x) = \lim_k u_{\mathcal{P}_k}(x) \geq H(x)$ provided $x \notin N$.

Proof

Proof of the Lemma Continued.

On the other hand, if $x \notin N$ and $\epsilon > 0$, there is a $\delta > 0$ such that

$$H(x) + \epsilon > \sup\{\, f(y) : |y - x| \le \delta \,\}$$

We can take k such that $\|\mathcal{P}_k\| < \frac{1}{k} < \delta$. Since $x \notin \mathcal{P}_k$, $x \in (t_{i-1}, t_i)$ for some interval determined by $\mathcal{P}_k$ and

$$M_i \le \sup\{\, f(y) : |y - x| \le \delta \,\}.$$

Then

$$H(x) + \epsilon > u_{\mathcal{P}_k}(x) \ge u(x).$$

Since $\epsilon > 0$ is arbitrary, we have shown that $H(x) = u(x)$ if $x \notin N$. Thus $H \sim u$ and H is measurable (since m is complete). Moreover,

$$\int_{[a,b]} H\, dm = \int_{[a,b]} u\, dm = \overline{\mathcal{R}}\int_a^b f.$$

The argument for h is similar. □

Proof

Proof of Part (2).

Suppose that f is continuous almost everywhere. Then by our first lemma, $H = h$ almost everywhere. Then our second lemma implies that the upper and lower Riemann integrals are equal. Hence $f \in \mathcal{R}[a, b]$ as claimed.

On the other hand, if $f \in \mathcal{R}[a, b]$, then the upper and lower Riemann integrals are equal. Using our second lemma, we have $H - h \geq 0$ and

$$\int_{[a,b]} (H - h)\, dm = 0.$$

Hence $H \sim h$ and by our first lemma, f is continuous almost everywhere. □

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Definition

Let $C_0 = [0,1]$, and let $C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$ be the closed set obtained by "removing the middle third" of C_0—that is $C_0 \setminus (\frac{1}{3},\frac{2}{3})$. Let $C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{5}{9}] \cup [\frac{8}{9},1]$ be the closed set obtained by removing the middle third of each of the two closed intervals in C_1. In general, for $n \geq 3$ let C_n be the union of the 2^n closed intervals of length $\frac{1}{3^n}$ obtained by removing the middle third of each of the closed intervals in C_{n-1}. Then the Cantor set is defined to be

$$\mathscr{C} = \bigcap_{n=1}^{\infty} C_n$$

What's in $\mathscr{C}$

Proposition

The Cantor set, $\mathscr{C}$ is an uncountable compact subset of $[0,1]$ with Lebesgue measure zero.

Proof.

Since $\mathscr{C}$ is the intersection of closed sets, it is closed in $[0,1]$ and therefore compact. Note that $m(C_n) = 2^n \cdot \frac{1}{3^n}$. Since $C_{n+1} \subset C_n$ and $m(C_1) = \frac{2}{3} < \infty$, $m(\mathscr{C}) = m\Bigl(\bigcap_n C_n\Bigr) = \lim_n m(C_n) = 0$. Since $\mathscr{C}$ is a closed subset of $\mathbf{R}$, it is a Baire space. Hence to show that $\mathscr{C}$ is uncountable, it will suffice to see that $\mathscr{C}$ has no isolated points. Let EC_n be the 2^{n+1} endpoints of the 2^n intervals making up C_n. Note that $EC_n \subset EC_{n+1}$. Hence $E = \bigcup_{n=0}^{\infty} EC_n \subset \mathscr{C}$. Let $x \in \mathscr{C}$ and let $r > 0$. Let n be such that $\frac{1}{3^n} < r$. Then x belongs to one of the intervals, I, making up C_n and then both endpoints of I are in $B_r(x)$. This means $\mathscr{C} \cap B_r(x) \setminus \{x\} \neq \emptyset$, and x. Since $r > 0$ is arbitrary, x is not isolated. □

Riemann Integrable

Remark

Let us suppose that the Cantor set $\mathscr{C}$ *contains a subset that is not Borel. Then if* τ *is the collection of open sets in* $\mathbf{R}$*, we have*

$$\tau \subsetneq \mathcal{B}(\mathbf{R}) \subsetneq \mathcal{L} \subsetneq \mathcal{P}(\mathbf{R}).$$

Let $f : \mathbf{R} \to \mathbf{R}$ *be a function. Then if* f *is continuous, it is necessarily Borel. If* f *is Borel, then it is necessarily Lebesgue measurable. However, the reverse implications all fail. It is interesting to ask where Riemann integrable functions sit in this hierarchy. But if* $A \subset \mathscr{C}$ *and* $A \in \mathcal{L} \setminus \mathcal{B}(\mathbf{R})$*, then since* $\mathscr{C}$ *is closed,* $f = \mathbb{1}_A$ *is continuous at all* $x \notin \mathbf{C}$*. Thus* f *is continuous almost everywhere and hence* $f \in \mathcal{R}[0,1]$*. But* f *is not Borel.*

Break Time

- Definitely time for a break.
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Completeness is Good

Remark

If $(X, \mathcal{M}, \mu)$ is a measure space, then we would like to put a natural norm on $\mathcal{L}^1(X, \mathcal{M}, \mu)$. The natural choice is

$$\|f\|_1 = \int_X |f(x)|\, d\mu(x) \tag{$*$}$$

as then $\|f_n - f\|_1 \to 0$ implies

$$\lim_n \int_X f_n(x)\, d\mu(x) = \int_X f(x)\, d\mu(x).$$

One small problem! In many cases—for example, for Lebesgue measure on $\mathbf{R}$*—($*$) is not a norm. We have $\|\alpha f\|_1 = |\alpha| \|f\|_1$ and $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$, but $\|f\|_1 = 0$ does not imply $f = 0$. Instead, we have $\|f\|_1 = 0$ if and only if $f = 0$ almost everywhere.*

Easy Solution

Definition

If $(X, \mathcal{M}, \mu)$ is a measure space, then we let $L^1(X, \mathcal{M}, \mu)$ be the set of almost everywhere equivalence classes in $\mathcal{L}^1(X, \mathcal{M}, \mu)$. We let $[f]$ be the class of $f \in \mathcal{L}^1(X)$ in $L^1(X)$.

Proposition

With respect to the operations $[f] + [g] = [f + g]$ *and* $\alpha[f] = [\alpha f]$, $L^1(X, \mathcal{M}, \mu)$ *is a complex vector space and*

$$\big\|[f]\big\|_1 = \|f\|_1$$

is a norm on $L^1(X)$.

Sketch of the Proof.

The proof simply amounts to observing that the above operations and definition of the norm are well-defined. □

Remark

Recall that a normed vector space $(V, \|\cdot\|)$ is complete, if (V, ρ) is complete in the induced metric $\rho(v,w) = \|v - w\|$. A complete normed vector space is called a Banach space.

Definition

If $(V, \|\cdot\|)$ is a normed vector space, then a series $\sum_{n=1}^{\infty} v_n$, with each $v_n \in V$, converges if the partial sums $s_n = v_1 + \cdots + v_n$ converge: that is, if there is a $v \in V$ such that $\|s_n - v\| \to 0$ with n. We say that $\sum_{k=1}^{\infty} v_n$ converges absolutely if $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

Remark (Be Careful)

Despite the terminology, there is nothing that says an absolutely convergent series is convergent.

Absolute Convergence

Proposition

A normed vectors space $(V, \|\cdot\|)$ is a Banach space (aka complete) if and only if every absolutely convergent series in V converges.

Proof.

Suppose that V is complete and $\sum_{n=1}^{\infty} \|v_n\| < \infty$. Let $s_n = v_1 + \cdots v_n$. We want to show that (s_n) is convergent. Since V is complete, it suffices to see that it is Cauchy. Let $\epsilon > 0$. Then there is a N such that $\sum_{k=N}^{\infty} \|v_k\| < \epsilon$. Then if $n \geq m \geq N$,

$$\|s_n - s_m\| = \Big\| \sum_{k=m+1}^{n} v_k \Big\| \leq \sum_{k=m+1}^{n} \|v_k\| \leq \sum_{k=N}^{\infty} \|v_k\| < \epsilon.$$

Therefore (s_n) converges as required.

Proof

Proof Continued.

Conversely, now suppose that absolutely convergent series are convergent. Let (v_n) be a Cauchy sequence in V. By HW#7, it will suffice to find a convergent subsequence of (v_n). Choose n_1 such that $n \geq n_1$ implies $\|v_n - v_{n_1}\| < \frac{1}{2}$. Choose $n_2 > n_1$ such that $n \geq n_2$ implies $\|v_n - v_{n_2}\| < \frac{1}{2^2}$. Notice that

$$\|v_{n_2} - v_{n_1}\| < \frac{1}{2}.$$

Continuing in this way, we find a subsequence (v_{n_k}) such that

$$\|v_{n_{k+1}} - v_{n_k}\| < \frac{1}{2^k}.$$

Proof

Proof Continued.

Let $g_1 = v_{n_1}$ and $g_k = v_{n_k} - v_{n_{k-1}}$ if $k \geq 2$. Then by construction

$$\sum_{k=1}^{\infty} \|g_k\| < \infty.$$

By assumption $\sum_{k=1}^{\infty} g_k$ is convergent. Therefore there is a $v \in V$ such that

$$v = \lim_{k\to\infty} \sum_{j=1}^{k} g_k = \lim_{k} v_{n_k}.$$

Thus (v_{n_k}) the convergent subsequence of (v_n) that we were looking for. □

Previews of Coming Attractions

Theorem

If $(X, \mathcal{M}, \mu)$ is a measure space, then $L^1(X, \mathcal{M}, \mu)$ is a Banach space.

Example

If ν is counting measure on $\mathbf{N}$, then $L^1(\mathbf{N}, \mathcal{P}(\mathbf{N}), \nu)$ is just ℓ^1, and we have already seen that ℓ^1 is complete. More generally, now we can define $\ell^1(X)$ to be $L^1(X, \mathcal{P}(X), \nu)$ for counting measure on any set X where $\|f\|_1 = \sum_{x \in X} |f(x)|$. Note that in this case,

$$\mathcal{L}^1(X, \mathcal{P}(X), \nu) = L^1(X, \mathcal{P}(X), \nu)$$

and every $f \in L^1(X, \nu)$ vanishes off a countable set.

That's Enough for Today

- That is enough for now.