Math 73/103: Fall 2020
Lecture 18

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Getting Started

- We should be recording!
- Questions?

$L^1(X)$ is a Banach Space

Theorem

If $(X, \mathcal{M}, \mu)$ is a measure space, then $L^1(X, \mathcal{M}, \mu)$ is a Banach space.

Remark

- *Recall that elements of $L^1(X)$ are actually almost everywhere equivalance classes $[f]$ for $f \in \mathcal{L}^1(X)$.*
- *Nevertheless, we almost always ignore this and work with bona fide function in $\mathcal{L}^1(X)$.*
- *For example, in the proof, we will take $\{ f_n \}_{n=1}^\infty \subset \mathcal{L}^1(X)$ such that $\sum_{n=1}^\infty \|f_n\|_1 < \infty$ and show that partial sums s_n converge "in $L^1(X)$" to some $s \in \mathcal{L}^1(X)$.*
- *This will suffice since $\|s_n - s\|_1 \to 0$ implies $\|[s_n] - [s]\|_1 = \|[s_n - s]\|_1 \to 0$ as required.*

Proof

Proof.

Suppose that $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$ for $f_n \in \mathcal{L}^1(X)$ as in the remark on the previous slide. Let

$$g(x) = \sum_{n=1}^{\infty} |f_n(x)| \in [0, \infty].$$

Then

$$\int_X g(x)\, d\mu(x) = \sum_{n=1}^{\infty} \int_X |f_n(x)|\, d\mu(x) = \sum_{n=1}^{\infty} \|f_n\|_1 < \infty.$$

This means $N = \{ x : g(x) = \infty \}$ must have measure zero.

Proof

Proof Continued.

Since $\mathbf{C}$ is complete, there is a $s(x) \in \mathbf{C}$ such that

$$s(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{if } x \notin N.$$

We can extend s to all of X by setting $s(x) = 0$ if $x \in N$. Now let

$$s_n = \sum_{k=1}^{n} \mathbb{1}_{X \setminus N} \cdot f_k.$$

Then each s_n is measurable and $s_n \to s$ pointwise. Moreover $|s_n(x)| \leq g(x)$ for all x. Hence the LDCT implies that

$$\int_X |s_n - s| \, d\mu = \|s_n - s\|_1 = 0.$$

This means that $\sum_{k=1}^{\infty} f_k = s$ in $L^1(X)$ as required. $\square$

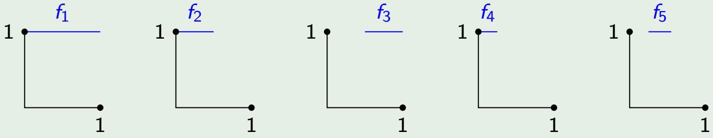
Convergence in L^1

Example

Let $f_n : [0,1] \to [0,1]$ be given by

$$f_n(x) = \mathbb{1}_{[\frac{j}{2^k}, \frac{j+1}{2^k}]}$$

where $n = 2^k + j$ with $0 \leq j < 2^k$.



Since $\int_0^1 f_{2^k+j}\, dm = 2^{-k}$, $f_n \to 0$ in $L^1([0,1])$. But $f_n \not\to 0$ almost everywhere. In fact, $(f_n(x))$ does not converge for any $x \in [0,1]$. But the subsequence $(f_{2^k}) \to 0$ for almost all x!

Convergence in Measure

Definition

Let $(X, \mathcal{M}, \mu)$ be a measure space. Then a sequence (f_n) of measurable functions from X to $\mathbf{C}$ converges in measure to a measurable function $f : X \to \mathbf{C}$ if for all $\epsilon > 0$ we have

$$\lim_n \mu\big(\{\, x : |f_n(x) - f(x)| \geq \epsilon \,\}\big) = 0.$$

Proposition

Suppose that $f_n \to f$ in $L^1(X, \mathcal{M}, \mu)$. Then $f_n \to f$ in measure.

Proof.

Let $E_n(\epsilon) = \{\, x : |f_n(x) - f(x)| \geq \epsilon \,\}$. Then $\|f_n - f\|_1 \geq \epsilon\mu(E_n(\epsilon))$. Since $\|f_n - f\|_1 \to 0$, for each $\epsilon > 0$ we must have $\mu(E_n(\epsilon)) \to 0$. This suffices. □

Why Do We Care

Theorem

Suppose $f_n \to f$ in measure. Then there is a subsequence (f_{n_k}) such that $f_{n_k} \to f$ almost everywhere.

Proof.

For each k, choose n_k such that $n \geq n_k$ implies

$$\mu(\{ x : |f_n(x) - f(x)| \geq 2^{-k} \}) < 2^{-k}.$$

Let $E_k = E_{n_k}(2^{-k}) := \{ x : |f_{n_k}(x) - f(x)| \geq 2^{-k} \}$. Let $G_k = \bigcup_{m \geq k} E_m$. Then by our choice of n_k, $\mu(G_k) \leq \sum_{m \geq k} \mu(E_k) \leq \sum_{m \geq k} 2^{-k} = 2^{-k+1}$. Suppose $x \notin G_k$. Then if $m \geq k$, we have $x \notin E_m$ so that $|f_{n_m}(x) - f(x)| < 2^{-m}$. This shows that $f_{n_m}(x) \to f(x)$ if $x \notin G_k$.

Proof Continued.

Now let

$$A = \bigcap_{k=1}^{\infty} G_k.$$

Note that if $x \notin A$, then $x \notin G_k$ for some k and $f_{n_m}(x) \to f(x)$. But $\mu(A) \leq \mu(G_k) \leq 2^{-k+1}$ for all k. Therefore $\mu(A) = 0$. □

Corollary

Suppose that $f_n \to f$ in $L^1(X, \mu)$. Then there is a subsequence (f_{n_k}) such that $f_{n_k} \to f$ almost everywhere.

Proof.

We know that $f_n \to f$ in measure. □

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

LDCT

Remark

The conclusion of the LDCT—namely that $\int_X |f_n - f| \, d\mu \to 0$*—can now be more elegantly stated as* $\|f_n - f\|_1 \to 0$ *or equivalently that* $f_n \to f$ *in* $L^1(X)$*. We can sharpen the LDCT a bit as follows.*

Theorem (LDCT revisited)

Suppose that $f_n \to f$ *in measure and that there is a* $g \in \mathcal{L}^1(X)$ *such that for each* n, $|f_n(x)| \leq g(x)$ *for almost all* x*. Then* $f_n \to f$ *in* $L^1(X)$*.*

Proof

Remark

For the proof, we need to observe that if $f_n \to f$ in measure, then any subsequence (f_{n_k}) also converges to f in measure. I leave you to verify this.

Proof.

Suppose that the conclusion of the theorem fails. Then there is a $\epsilon_0 > 0$ and a subsequence (f_{n_k}) such that

$$\|f_{n_k} - f\|_1 \geq \epsilon_0 \quad \text{for all } k. \tag{†}$$

Since $f_{n_k} \to f$ in measure, there is a subsubsequence $(f_{n_{k_j}})$ such that $f_{n_{k_j}} \to f$ almost everywhere. But then our old LDCT (almost everywhere version) implies that $\|f_{n_{k_j}} - f\|_1 \to 0$. This contradicts (†). □

Break Time

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Littlewood's Three Principles

Remark

Littlewood was a prominent analyst at Cambridge in the early part of the 20^{th} century. In his 1944 "Lectures on the Theory of Functions" he opined that the Lebesgue theory was not so mysterious because

> *"[t]here are three principles, roughly expressible in the following terms: Every [Lebesgue measurable] set is nearly a finite sum of intervals; every [Lebesgue measurable] function ... is nearly continuous; every [pointwise] convergent sequence of [measurable] functions is nearly uniformly convergent."*

Theorem (Littewood's First Principle)

If $E \subset \mathbf{R}$ has finite Lebesgue measure, then for all $\epsilon > 0$ there is a finite disjoint union of intervals F such that $m(E \Delta F) < \epsilon$ where $E \Delta F$ is the symmetric difference $(E \setminus F) \cup (F \setminus E)$.

This is HW#38 based on HW#37.

Egoroff's Theorem

Remark

As an illustration of Littlewood's third principal—that every pointwise convergent sequence is nearly uniformly convergeent—we have Egoroff's Theorem. This seems to be named after a Russian mathematician "Egorov" but I have always seen it spelled "Egoroff".

Theorem (Egoroff's Theorem)

Suppose that $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < \infty$. (We say that $(X, \mathcal{M}, \mu)$ is a finite measure space.) Suppose that $f_n : X \to \mathbf{C}$ is measurable for all $n \in \mathbf{N}$ and that $f_n \to f$ pointwise almost everywhere. Assuming that f itself is measurable, then for all $\epsilon > 0$ there is set $E \in \mathcal{M}$ such that $\mu(E) < \epsilon$ and such that $f_n \to f$ uniformly on $X \setminus E$.

Proof

Proof.

We may as well assume that $f_n(x) \to f(x)$ for all $x \in X$. For each $n, k \in \mathbf{N}$, let

$$E_n(k) = \bigcup_{m=n}^{\infty} \{ x : |f_m(x) - f(x)| \geq \frac{1}{k} \}.$$

Notice that if $x \notin E_n(k)$, then

$$|f_m(x) - f(x)| < \frac{1}{k} \quad \text{for all } m \geq n.$$

Furthermore, $E_{n+1}(k) \subset E_n(k)$ and $\bigcap_{n=1}^{\infty} E_n(k) = \emptyset$. Since $\mu(X) < \infty$, we have $\lim_n \mu\big(E_n(k)\big) = 0$.

Proof

Proof Continued.

Fix $\epsilon > 0$. Let n_k be such that $\mu\big(E_{n_k}(k)\big) < \frac{\epsilon}{2^k}$. Let $E = \bigcup_{k=1}^{\infty} E_{n_k}(k)$. Then $\mu(E) < \epsilon$. Furthermore, if $x \notin E$, then

$$|f_n(x) - f(x)| < \frac{1}{k} \quad \text{for all } n > n_k.$$

Therefore $f_n \to f$ uniformly on $X \setminus E$. □

More About $L^1(X)$

Remark

To see an example of Littlewood's second principle, we will prove a version of Lusin's Theorem.

Theorem (Lusin's Theorem)

Suppose that $f : [a, b] \to \mathbf{C}$ is Lebesgue measurable. Given $\epsilon > 0$ there is a closed subset $K \subset [a, b]$ such that $m\big([a, b] \setminus K\big) < \epsilon$ and $f|_K$ is continuous.

Remark

Note that Lusin's Theorem does not say that f need be continuous anywhere! Consider $f = \mathbb{1}_{\mathbf{Q}\cap[a,b]}$. Then f is not continuous at a single point, but $f|_{[a,b]\setminus\mathbf{Q}}$ is constant and therefore continuous.
Before we prove Lusin's Theorem, we need a bit more information about $L^1(X)$.

Simple Functions Again

Proposition

Let $(X, \mathcal{M}, \mu)$ be a measure space. Then the collection of integrable simple functions in $L^1(X, \mathcal{M}, \mu)$ is dense.

Proof.

Let f be an element of $\mathcal{L}^1(X)$. Given $\epsilon > 0$, it suffices to find a simple function $s \in \mathcal{L}^1(X)$ such that $\|s - f\|_1 < \epsilon$. Suppose $f : X \to [0, \infty)$. Then there are MNNSFs (s_n) such that $s_n \nearrow f$. In particular, each $s_n \leq f$ and belongs to $\mathcal{L}^1(X)$. (In fact, each s_n is a finite linear combinations of characteristic functions of sets of finite measure.) By the MCT (or the LDCT), $\|s_n - f\|_1 \to 0$.

In general, $f = u^+ - u^- + i(v^+ - v^-)$ with $\{ u^\pm, v^\pm \} \subset \mathcal{L}^1(X)$. Hence each of $\{ u^\pm, v^\pm \} \subset \mathcal{L}^1(X)$ can be approximated with MNNSFs. Since the collection of integrable simple functions is a subspace of $\mathcal{L}^1(X)$, the result follows. □

Step Functions

Definition

A function $s : \mathbf{R} \to \mathbf{C}$ is called a step function if it can expressed as a finite linear combination of characteristic functions of intervals. That is,

$$s = \sum_{k=1}^{n} \alpha_n \mathbb{1}_{I_k}$$

where each $I_k \subset \mathbf{R}$ is an interval.

Example

Let $\mathcal{P} = \{ a = t_0 < \cdots < t_n = b \}$ be a partition of $[a, b]$. Then

$$s = \sum_{k=1}^{n} \alpha_k \mathbb{1}_{[t_{k-1}, t_k)}$$

is a step function.

Step Functions are Dense

Proposition

The collection of step functions in $L^1(\mathbf{R}, m)$ is dense.

Proof.

Since simple functions are always dense, it suffices to show that we can approximate an integrable simple function s. If we write

$$s = \sum_{k=1}^{n} \alpha_k \mathbb{1}_{E_k},$$

in standard form, then, since $s \in \mathcal{L}^1(\mathbf{R})$, we must have $m(E_k) < \infty$ for each k. Hence is suffices to show that we can approximate $\mathbb{1}_E$ with $m(E) < \infty$. But given $\epsilon > 0$, we can find a disjoint union of (open) intervals F such that $m(E \Delta F) < \epsilon$. But then $\mathbb{1}_F$ is a step function and

$$\|\mathbb{1}_F - \mathbb{1}_E\|_1 = m(F \Delta E) < \epsilon. \quad \square$$

That's Enough for Today

- That is enough for now.