Math 73/103: Fall 2020
Lecture 19

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Getting Started

- We should be recording!
- Questions?

Continuous Functions in $\mathcal{L}^1(\mathbf{R})$

Proposition

Suppose that $f \in \mathcal{L}^1(\mathbf{R}, \mathcal{L}, m)$. Then given $\epsilon > 0$ there is a continuous function $g : \mathbf{R} \to \mathbf{C}$ such that g vanishes off a closed bounded interval and such that $\|g - f\|_1 < \epsilon$.

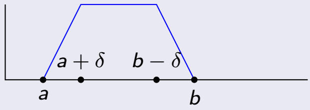
Remark

A continuous function $g : \mathbf{R} \to \mathbf{C}$ is said to have compact support if there is a $n \in \mathbf{N}$ such that $g(x) = 0$ if $x \notin [-n, n]$. The collection $C_c(\mathbf{R})$ of continuous compactly supported functions on $\mathbf{R}$ is a (vector space) subspace of $C(\mathbf{R})$. The map $f \mapsto [f]$ is an injection of $C_c(\mathbf{R})$ onto a subspace of $L^1(\mathbf{R})$ which we normally identify with $C_c(\mathbf{R})$. Another way to state the above proposition is that $C_c(\mathbf{R})$ is dense in $L^1(\mathbf{R})$. Since $\|\cdot\|_1$ is really a norm on $C_c(\mathbf{R})$, we can view $L^1(\mathbf{R})$ as the completion of $C_c(\mathbf{R})$ with respect to the metric $\rho(f, g) = \|f - g\|_1$.

Proof

Proof.

On Friday, we proved that given $\delta > 0$ there is a step function $s = \sum_{k=1}^{n} \alpha_k \mathbb{1}_{I_k}$ such that $\|f - s\|_1 < \delta$ where I_k is a bounded open interval. Since there are only finitely many I_k, there is a n such that $I_k \subset [-n, n]$ for all k. Therefore it will suffice to show that if $(a, b) \subset [-n, n]$ then there is a continuous function g vanishing off $[-n, n]$ approximating $\mathbb{1}_{(a,b)}$. But this is routine:



Here $\|g - \mathbb{1}_{(a,b)}\|_1 = \delta$. □

Lusin's Theorem

Theorem (Lusin's Theorem)

Suppose that $f : [a,b] \to \mathbf{C}$ is Lebesgue measurable. Given $\epsilon > 0$ there is a closed subset $K \subset [a,b]$ such that $m\big([a,b] \setminus K\big) < \epsilon$ and $f|_K$ is continuous.

Proof.

Fix $\epsilon > 0$. Let $A_n = \{\, x \in [a,b] : |f(x)| \leq n \,\}$. Then $\bigcup A_n = [a,b]$. Since $A_n \subset A_{n+1}$, $\lim_n m(A_n) = m([a,b]) = b - a$ and there is a n such that $m([a,b] \setminus A_n) < \frac{\epsilon}{3}$. Let

$$h(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq n \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

Since h is bounded, $h \in \mathcal{L}^1([a,b])$. Since $C([a,b])$ is dense in $L^1([a,b])$ there is a sequence $(g_n) \subset C([a,b])$ such that $g_n \to h$ in $L^1([a,b])$.

Proof Continued.

Then there is a subsequence (g_{n_k}) such that $g_{n_k} \to h$ almost everywhere. By Egoroff's Theorem, there is a set E such that $m([a,b] \setminus E) < \frac{\epsilon}{3}$ and such that $g_{n_k} \to h$ uniformly on E. Thus $h|_E$ is continuous. Check that $m([a,b] \setminus (E \cap A_n)) < \frac{2\epsilon}{3}$. Using HW#37, there is a closed set $K \subset E \cap A_n$ such that $m((E \cap A_n) \setminus K) < \frac{\epsilon}{3}$. Now you can confirm that $m([a,b] \setminus K) < \epsilon$ as required and $h|_K$ is continuous. □

Outside the Lines

Remark

If $K \subset [a, b]$ is closed and $h : K \to \mathbf{R}$ is continuous, then in another course we might have time to prove that there is a continuous function $g : [a, b] \to \mathbf{R}$ that extends h; that is, $g(x) = h(x)$ for all $x \in K$. This is called the Tietze Extension Theorem. Assuming this, we can re-cast Lusin's Theorem as follows.

Corollary

Suppose that $f : [a, b] \to \mathbf{C}$ is Lebesgue measurable and that $\epsilon > 0$. Then there is a continuous function $g : [a, b] \to \mathbf{C}$ such that $m\big(\{ x \in [a, b] : g(x) \neq f(x) \}\big) < \epsilon$.

Proof.

By our version of Lusin's Theorem, there is a closed set $K \subset [a, b]$ such that $f|_K$ is continuous and $m([a, b] \setminus K) < \epsilon$. By Tietze, we can let g be an extension of $h = f|_K$ to all of $[a, b]$. □

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Complex Measures

Definition

A complex measure on a measurable space $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to \mathbf{C}$ such that $\nu(\emptyset) = 0$ and

$$\nu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \nu(E_n) \qquad (\dagger)$$

whenever each $E_n \in \mathcal{M}$ and $E_n \cap E_m = \emptyset$ if $n \neq m$. If $\nu(\mathcal{M}) \subset (-\infty, \infty)$, then we call ν a real-valued measure.

Remark

Royden & Fitzpatrick use the term "signed measure" in place of a "real-valued measure". Even for real-valued measures, the values $\pm\infty$ are not allowed. Since the left-hand side of $(\dagger)$ is invariant under re-arrangement, the right-hand side is as well. This forces the convergence of the series on the right-hand side to be absolute.

Examples

Example

Let μ_k for $k = 1, 2, 3, 4$ be finite measures on $(X, \mathcal{M})$. Then

$$\nu(E) = \mu_1(E) - \mu_2(E) + i\big(\mu_3(E) - \mu_4(E)\big)$$

is a complex measure on $(X, \mathcal{M})$. In my mind, it will be a little disappointing to work quite hard to show that all examples arise in this way. Nevertheless, this discovery will prove very useful.

Agreements and Conventions

Remark

1. *Since a complex measure is clearly the sum of two real-valued measures, we will concentrate almost exclusively on real-valued measures.*
2. *Royden & Fitzpatrick, Rudin, and Folland all allow real-valued measures to take either the value ∞ or the value $-\infty$ (but not both). There are good reasons for this, but we will settle for the simpler path.*
3. *When we refer to simply a "measure" we always mean a good old fashioned set function taking values in $[0, \infty]$. If we feel the need to be pedantic, then we might say "positive measure".*
4. *It is an unfortunate result of this terminology that a real-valued measure or a complex measure that also happens to be a positive measure must be a finite measure.*

Our Intuition Needs to be Upgraded

Definition

Let ν be a real-valued measure on $(X, \mathcal{M})$. Then we say $P \in \mathcal{M}$ is positive if $\nu(E) \geq 0$ for all measureable subsets $E \subset P$. Similarly, we say that $N \in \mathcal{M}$ is negative if $\nu(E) \leq 0$ for all measurable subsets $E \subset N$. A set N is a null set if it is both positive and negative.

Remark

Notice first that $\nu(E) = 0$ does not imply that N is a null set! If these definitions seem overly fussy, consider $([-1,1], \mathcal{L}([-1,1]))$ and $\nu(E) = m(E \cap [-1,0]) - m(E \cap [0,1])$ where m is Lebesgue measure. Then $\nu([-1,1]) = 0$, but it does not seem proper to dismiss $[-1,1]$ as a "null set".

Lemma

Every measurable subset of a positive set is positive as is the countable union of positive sets.

Proof.

The assertion about subsets is clear. Suppose $P = \bigcup P_n$ with each P_n positive. We can assume the P_n are pairwise disjoint. (Why?) Then if $E \subset P$,

$$\nu(E) = \nu\Big(\bigcup E \cap P_n\Big) = \sum_n \nu(E \cap P_n) \geq 0. \quad \square$$

Do Positive Sets Even Exist?

Proposition

Suppose that ν is a real-valued measure on $(X, \mathcal{M})$ and that $\nu(E) > 0$. Then E contains a positive set P with $\nu(P) > 0$.

Proof.

If E is positive, we're done. Otherwise let n_1 be the smallest positive integer such that there is a $E_1 \subset E$ with

$$\nu(E_1) \leq -\frac{1}{n_1}.$$

We proceed inductively. Suppose we have picked disjoint subsets $E_1, E_2, \dots, E_{k-1}$. If $E \setminus \bigcup_{j=1}^{k-1} E_j$ is not positive, then we can let n_k be the smallest positive integer such that there is an $E_k \subset E \setminus \bigcup_{j=1}^{k-1} E_j$ with

$$\nu(E_k) \leq -\frac{1}{n_k}.$$

Note that $E_1, \dots, E_k$ are pairwise disjoint.

Proof

Proof Continued.

I claim that if $E \setminus \bigcup_{j=1}^{k-1} E_j$ is positive, then we are done:

$$\begin{aligned} 0 < \nu(E) &= \nu\Big(E \setminus \bigcup_{j=1}^{k-1} E_j\Big) + \nu\Big(\bigcup_{j=1}^{k-1} E_j\Big) \\ &= \nu\Big(E \setminus \bigcup_{j=1}^{k-1} E_j\Big) + \sum_{j=1}^{k-1} \nu(E_j) \leq \nu\Big(E \setminus \bigcup_{j=1}^{k-1} E_j\Big). \end{aligned}$$

If this process does not terminate with a positive set, then let

$$A = E \setminus \bigcup_{j=1}^{\infty} E_j.$$

Proof Continued.

Just as on the last slide,

$$0 < \nu(E) = \nu(A) + \sum_{j=1}^{\infty} \nu(E_j) \leq \nu(A)$$

and $\nu(A) > 0$. Thus it will suffice to see that A is positive. Notice that

$$\sum_{j=1}^{\infty} \nu(E_j) = \nu\Big(\bigcup_{j=1}^{\infty} E_j\Big) \in (-\infty, 0].$$

This means

$$-\infty < \sum_{j=1}^{\infty} \nu(E_j) \leq -\sum_{j=1}^{\infty} \frac{1}{n_j}.$$

Proof

Proof Continued.

Therefore $\sum \frac{1}{n_j} < \infty$ and $n_j \to \infty$. Fix $\epsilon > 0$. Then there is a k such that $\frac{1}{n_k - 1} < \epsilon$. Then

$$A \subset E \setminus \bigcup_{j=1}^{k-1} E_j$$

can contain no measurable subset F such that

$$\nu(F) \leq -\frac{1}{n_k - 1} > -\epsilon.$$

Thus for all $F \subset A$, $\nu(F) > -\epsilon$. Since ϵ was arbitrary, we shown that A is positive. □

Break Time

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Theorem (Hahn Decomposition)

Let ν be a real-valued measure on $(X, \mathcal{M})$. Then there is a partition $X = P \cup N$ such that P is postive and N is negative. If $P' \cup N'$ is another such decomposition, then $P \Delta P'$ and $N \Delta N'$ are null sets.

Proof.

Let $\mathscr{P}$ be the collection of all positive sets in X. Note that $\emptyset \in \mathscr{P}$. Let

$$\lambda = \sup\{\, \nu(A) : A \in \mathscr{P} \,\} \in [0, \infty].$$

Let $A_n \in \mathscr{P}$ be such that $\nu(A_n) \to \lambda$. Let $P = \bigcup_{n=1}^{\infty} A_n$. Then $P \in \mathscr{P}$ and $\nu(P) \leq \lambda$. But $P \setminus A_n \in \mathscr{P}$ and

$$\nu(P) = \nu(A_n) + \nu(P \setminus A_n) \geq \nu(A_n).$$

Therefore $\nu(P) = \lambda$ and $\lambda < \infty$.

Proof

Proof Continued.

Let $N = X \setminus P$. Suppose that $E \subset N$ is such that $\nu(E) > 0$. Then there is a positive set $A \subset E \subset N$ such that $\nu(A) > 0$. But then $P \cup A \in \mathscr{P}$ and $\nu(P \cup A) > \lambda$. This is a contradiction, so we conclude that $\nu(E) \leq 0$ for all measurable $E \subset N$. That is, N is negative.

Uniqueness up to null sets is left for homework. □

Hahn Decomposition

Definition

The partition $\{ P, N \}$ in the previous result is called a Hahn Decomposition for ν.

Remark

Note that a Hahn decomposition for ν is unique up to null sets.

Definition

Two (positive) measures are mutually singular—written $\mu_1 \perp \mu_2$—if there is a partition $X = A \cup B$ such that $\mu_1(B) = 0 = \mu_2(A)$.

Jordan Decomposition

Theorem (Jordan Decomposition)

Let ν be a real-valued measure on $(X, \mathcal{M})$. Then there is a unique pair of mutually singular finite measures ν^+ and ν^- on (X, M) such that $\nu = \nu^+ - \nu^-$. We call this a Jordan decomposition for ν.

Proof.

Let $\{ P, N \}$ be a Hahn decomposition for ν. Then we can set $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$. The rest is straightforward. □

That's Enough for Today

- That is enough for now.