Math 73/103: Fall 2020
Lecture 2

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Getting Started

- We should be recording!
- Does anyone have any questions or comments about the lecture or the homework?
- As I mentioned, I hope that you have the bandwidth to keep your video on during the class meeting. This makes it seem a little more "real" for me. But this is voluntary.

Subspaces

Proposition

Let (X, ρ) be a metric space and $E \subset X$ a subspace. Then $U \subset E$ is open in E if and only if there is an open set V in X such that $U = V \cap E$.

Proof.

As usual, let $B_r(x) = \{ y \in X : \rho(y, x) < r \}$ and write $B_r^E(x) = \{ y \in E : \rho(y, x) < r \} = B_r(x) \cap E$.

Now suppose V is open in X and $U = V \cap E$. Suppose $x \in U$. Then there is a $r > 0$ such that $B_r(x) \subset V$. Then of course $B_r^E(x) = B_r(x) \cap E \subset V \cap E = U$. Hence U is open in E.

The Converse

Proof.

Now suppose that U is open in E. If $x \in U$, then there is a $r(x) > 0$ such that $B^E_{r(x)}(x) = B_{r(x)}(x) \cap E \subset U$. Let $V = \bigcup_{x \in U} B_{r(x)}(x)$. Then V is open in X and

$$V \cap E = \Big(\bigcup_{x \in U} B_{r(x)}(x) \Big) \cap E = \bigcup_{x \in U} B^E_{r(x)}(x) \subset U.$$

But if $x \in U$, then $x \in B_{r(x)}(x) \subset V$. Hence $U \subset V \cap E$ and $U = V \cap E$ as required. □

Corollary

If E is a subspace of (X, ρ), then then the metric topology, τ_E on E is

$$\tau_E = \{ V \cap E : V \in \tau_\rho \}$$

where τ_ρ is the metric topology on X.

Closed Sets

Definition

Let (X, ρ) be a metric space. Then $F \subset X$ is closed if $F^c := X \setminus F$ is open.

Proposition

Let (X, ρ) be a metric space.

1. *Both X and $\emptyset$ are closed.*
2. *If F and G are closed, then so is $F \cup G$.*
3. *If F_a is closed for all $a \in A$, then $\bigcap_{a \in A} F_a$ is also closed.*

Proof.

This is straightforward. □

Closure

Definition

If E is a subset of metric space X, then

$$\overline{E} := \bigcap \{\, F \subset X : F \text{ is closed and } E \subset F \,\}$$

is called the closure of E in X.

Remark

Note that $\overline{E}$ is the smallest closed set containing E.

Closure

Lemma

We have $x \in \overline{E}$ if and only if $B_r(x) \cap E \neq \emptyset$ for all $r > 0$.

Proof.

Suppose that $x \in \overline{E}$. Suppose to the contrary that there is a $r > 0$ such that $B_r(x) \cap E = \emptyset$. Let $F = X \setminus B_r(x)$. Then F is closed and $E \subset F$. Therefore $\overline{E} \subset F$ and $x \notin \overline{E}$ which is a contradiction. Now suppose that $B_r(x) \cap E \neq \emptyset$ for all $r > 0$. Suppose to the contrary that $x \notin \overline{E}$. Then x is an element of the open set $X \setminus \overline{E}$. Hence there is a $r > 0$ such that $B_r(x) \subset X \setminus E$. But then $B_r(x) \cap E = \emptyset$. □

Break Time

- Time to take a Break.
- But first, let's see if there are questions or comments.

- We should be recording.

Definition

Let (X, ρ) be a metric space and suppose that (x_n) is a sequence in X. We say that (x_n) converges to $x \in X$ if for all $\epsilon > 0$ there is a $N \in \mathbf{N}$ such that $n \geq N$ implies $\rho(x_n, x) < \epsilon$. If (x_n) converges to x, then we write $\lim_{n\to\infty} x_n = x$ or sometimes $x_n \to x$.

Lemma

Limits, if they exist, are unique. That is if $x_n \to x$ and $x_n \to y$, then $x = y$.

Proof.

If $x \neq y$, then let $\epsilon = \frac{1}{2}\rho(x, y)$ and find a contradiction. □

Subsequences

Definition

If (x_n) is a sequence in X and if $n_1 < n_2 < \cdots$ forms an infinite subset of $\mathbf{N}$, then the sequence (y_k), where $y_k = x_{n_k}$, is called a subsequence if (x_n).

Remark

Normally, we simply write (x_{n_k}) and dispense with introducing y_k. Note that if (x_{n_k}) is a subsequence, then $n_k \geq k$ for all k. Then it is easy to prove that if if $x_n \to x$, then any subsequence of (x_n) also converges to x.

Example

Example

Suppose that X is a metric space and (x_n) is a sequence that does not converge to x. Then there is a $r > 0$ and a subsequence (x_{n_k}) such that $x_{n_k} \notin B_r(x)$ for all k.

Proof.

There must be a $r > 0$ so that for all $N \in \mathbf{N}$, the statement $n \geq N$ implies $x_n \in B_r(x)$ is false. Hence there is some n_1 such that $x_{n_1} \notin B_r(x)$. Now suppose we have found $n_1 < n_2 < \cdots < n_k$ such that $x_{n_j} \notin B_r(x)$ for all $1 \leq j \leq k$. Since $n \geq n_k$ does not imply $x_n \in B_r(x)$, there is a $n_{k+1} > n_k$ such that $x_{n_{k+1}} \notin B_r(x)$. This produces the required subsequence (x_{n_k}). □

Limit Points

Definition

If E is a subset of a metric space X, then $x \in X$ is a limit point of E if there is a sequence $(x_n) \subset E$ such that $x_n \to x$ in X.

Proposition

A subspace E of a metric space X is closed if and only if it contains all of its limit points.

Proof.

I'm leaving this as homework. I also asked you to formulate and prove the appropriate sequential characterization of open sets! □

Equivalent Metrics

Definition

Suppose that ρ and σ are metrics on X. Then we say ρ and σ are equivalent if they generate the same metric topology.

Lemma

Two metrics ρ and σ on X are equivalent if for all $x \in X$ and all $r > 0$ there are $r', r'' > 0$ such that

1. $B^{\rho}_{r'}(x) \subset B^{\sigma}_{r}(x)$ *and*
2. $B^{\sigma}_{r''}(x) \subset B^{\rho}_{r}(x)$.

Proof.

Using (1), you can show $U \in \tau_\sigma$ implies $U \in \tau_\rho$. Using (2), it follows that $U \in \tau_\rho$ implies $U \in \tau_\sigma$. □

Bounded Metrics

Lemma

Let (X, ρ) be a metric space. Then $d(x, y) := \dfrac{\rho(x, y)}{1 + \rho(x, y)}$ is an equivalent metric on X.

Proof.

We've already seen that d is a metric. Also it is immediate that $d(x, y) \leq \rho(x, y)$. Hence for all $x \in X$, $B_r^\rho(x) \subset B_r^d(x)$.
On the other hand, if $\rho(x, y) \leq 1$, then $\rho(x, y) \leq 2d(x, y)$. Let $r' = \min\{ 1, r \}$. Then if $y \in B_{r'/2}^d(x)$, we have $d(y, x) < \frac{r'}{2} \leq \frac{1}{2}$.
A little algebra implies that $\rho(y, x) < 1$. Then $\rho(y, x) \leq 2d(y, x) < r' \leq r$. That is, $B_{r'/2}^d(x) \subset B_r^\rho(x)$. □

Other Bounded Metrics

Remark

The previous result shows every metric space X admits an equivalent bounded metric—bounded by 1 in fact. There is nothing special about $d(x, y) = \rho(x, y)/(1 + \rho(x, y))$. You are welcome to prove a similar result for

$$\overline{\rho}(x, y) = \min\{\, 1, \rho(x, y)\,\}.$$

That is, you can show that $\overline{\rho}$ is a metric on X and that it is equivalent to ρ.

Topology vs Metric

Definition

If E is a subspace of (X, ρ), then the diameter of E is

$$\operatorname{diam}(E) = \sup\{\, \rho(x, y) : x, y \in E \,\}.$$

We say that E is bounded if $\operatorname{diam}(E) < \infty$.

Remark

If ρ and σ are strongly equivalent metrics on X, then $E \subset X$ is bounded with respect to ρ if and only if it is bounded with respect to σ. However, this need not be the case for equivalent metrics. Note that $\mathbf{R}$ is unbounded with respect to its usual metric, but it is bounded with respect to the equivalent metric $d(x, y) = \dfrac{|x - y|}{1 + |x - y|}$.

Continuity

Definition

Let $f : (X, \rho) \to (Y, \sigma)$ be a function. We say that f is continuous at $x_0 \in X$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$f\big(B_\delta^\rho(x_0)\big) \subset B_\epsilon^\sigma\big(f(x_0)\big). \tag{1}$$

If f is continuous at all $x_0 \in X$, then we just say f is continuous.

Remark

Condition (1) is equivalent to $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\rho(x, x_0) < \delta \implies \sigma\big(f(x), f(x_0)\big) < \epsilon$.

The First Question

Proposition

Let $f : (X, \rho) \to (Y, \sigma)$ be a function. Then the following are equivalent.

1. *f is continuous at $x_0 \in X$.*
2. *If (x_n) is any sequence converging to x_0 in X, then $f(x_n) \to f(x_0)$.*
3. *If V is any neighborhood of $f(x_0)$ in Y, then $f^{-1}(V)$ is a neighborhood of x_0 in X.*

Proof.

You should be able to give a proof of this off the top of your head or agree never to teach calculus! The only tricky bit is observing that if $f^{-1}(V)$ is not a neighborhood of x_0, then there must be a sequence (x_n) converging to x_0 in X with $x_n \notin f^{-1}(V)$ for all n. □

Continuous Functions

Corollary

A function $f : (X, \rho) \to (Y, \sigma)$ is continuous if and only if $f^{-1}(V)$ is open in X for every open set $V \subset Y$.

Definition

We say that $f : (X, \rho) \to (Y, \sigma)$ is uniformly continuous on X if for all $\epsilon > 0$ there is a $\delta > 0$ such that $\rho(x, y) < \delta$ implies $\sigma\big(f(x), f(y)\big) < \epsilon$.

Remark

If f is uniformly continuous, then it is continuous: given any $x_0 \in X$ and any $\epsilon > 0$ we can choose a δ—depending only on ϵ and not on x_0—such that $\rho(x, x_0) < \delta$ implies $\sigma\big(f(x), f(x_0)\big) < \epsilon$. However, if we only assume f is continuous, then our choice of δ may depend on the choice of x_0—that is, $\delta = \delta(\epsilon, x_0)$—and we may not be able to find a δ that works for all x_0 at the same time. An example is $f(x) = x^2$ from $\mathbf{R}$ to $\mathbf{R}$.

Break Time

- Time for another recording Break.
- First, questions or comments?
- Ok, restart the recording.

Completeness

Definition

We say that a sequence (x_n) in a metric space (X, ρ) is Cauchy if for all $\epsilon > 0$ there is a $N \in \mathbf{N}$ such that $n, m \geq N$ implies $\rho(x_n, x_m) < \epsilon$.

Example

If (x_n) converges in X, then (x_n) is cauchy: Given $\epsilon > 0$, there is a N such that $n \geq N$ implies $\rho(x_n, x_0) < \epsilon/2$. But then if $n, m \geq N$, we have $\rho(x_n, x_m) \leq \rho(x_n, x_0) + \rho(x_0, x_m) < \epsilon$.

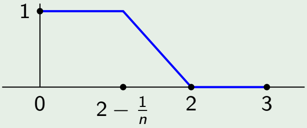
Not so Fast

Example

Let $X = (0,2)$ with the usual metric. If $x_n = \frac{1}{n}$, then (x_n) is Cauchy in X, but does not converge in X.

Example

Of course the problem in the previous example is that some idiot took 0 away from us. But in general, the "missing point" might be more subtle. Let $X = C([0,3])$ equipped with the metric ρ associated to the norm $\|f\|_1 = \int_0^3 |f(t)|\,dt$. Let f_n be the function with graph



I leave it to you to check that (f_n) is Cauchy in (X,ρ) and that (f_n) does not converge in X.

Complete Spaces

Definition

A metric space is complete if every Cauchy sequence in X is convergent.

Lemma (Cauchy Sequences Want to Converge)

If a Cauchy sequence (x_n) *has a convergent subsequence, then it converges.*

Proof.

I leave this as homework exercise. □

Example

Both **R** and **C** are complete with respect to their usual metrics.

The ℓ^p Spaces are Complete

Theorem

For all $1 \leq p \leq \infty$, ℓ^p is complete (in the metric induced from $\|\cdot\|_p$). Furthermore, $\ell^\infty(X)$ is complete for any set X.

Proof.

To start with let $1 \leq p < \infty$ and suppose that (x_n) is a Cauchy sequence in ℓ^p. Since x_n is itself a sequence, we will write $x_n(k)$ for the k^{th}-term of x_n. Note that for each k, $|x_n(k)| \leq \|x_n\|_p = \left(\sum_{k=1}^\infty |x_n(k)|^p\right)^{\frac{1}{p}}$. In particular, $(x_n(k))_{n=1}^\infty$ is Cauchy in $\mathbf{C}$ for all k. Hence we can let $x_0(k) = \lim_{n\to\infty} x_n(k)$.

To finish the proof we need to establish that $x_0 \in \ell^p$ and that $x_n \to x_0$ in ℓ^p. Since $\rho(x_n, x_0) = \|x_n - x_0\|_p$, this means showing $\|x_n - x_0\|_p \to 0$ with n.

The Proof

Continued.

Fix $\epsilon > 0$. Let N be such that $n, m \geq N$ implies $\rho(x_n, x_m) = \|x_n - x_m\|_p < \frac{\epsilon}{2}$. Let $n \geq N$. Then for any $r \in \mathbf{N}$,

$$\sum_{k=1}^{r} \big|x_n(k) - x_0(k)\big|^p = \lim_{m\to\infty} \sum_{k=1}^{r} \big|x_n(k) - x_m(k)\big|^p$$
$$\leq \limsup_{m\to\infty} \|x_n - x_m\|_p^p \leq \Big(\frac{\epsilon}{2}\Big)^p.$$

Since this holds for any r, we see that $\|x_n - x_0\|_p \leq \frac{\epsilon}{2} < \epsilon$. This also implies that $x_n - x_0 \in \ell^p$. Since ℓ^p is a vector space, $\|x_0\|_p \leq \|x_0 - x_N\|_p + \|x_N\|_p \leq \epsilon + \|x_N\|_p < \infty$, and $x_0 \in \ell^p$.

Now the above shows that $x_n \to x_0$ in ℓ^p and we're done when $1 \leq p < \infty$. The proof for $\ell^\infty(X)$ is similar, but quite a bit easier. I leave that as an exercise. □

Low Hanging Friut

Proposition

Let X be a complete metric space. Then a subspace $E \subset X$ is complete if and only if E is closed.

Proof.

Suppose that E is closed. Let (x_n) be a Cauchy sequence in E. Then (x_n) is Cauchy in X and hence converges to some $x_0 \in X$. But E is closed if and only if it contains all its limit points. Hence $x_0 \in E$ since E is closed. Therefore (x_n) converges in E, and E is complete.

Now suppose that E is complete and $x_0 \in X$ is a limit point for E. Suppose that (x_n) is a sequence in E that converges to some x_0. Then (x_n) is Cauchy in E. Hence (x_n) converges to some $y \in E$. But limits are unique, so $x_0 = y \in E$. Therefore E contains all its limit points and is closed. □

$C(X)$

Notation

If X is a metric space, then we will write $C(X)$ for the complex vector-space of continuous complex-valued functions on X. Let also let $C_b(X)$ be the vector subspace of bounded functions in $C(X)$.

Remark

We can view $C_b(X)$ as a (vector) subspace of $\ell^\infty(X)$. Hence it inherits the supremun norm which induces the subspace metric. Note that (f_n) converges to f in $C_b(X)$ or $\ell^\infty(X)$ if and only if $\rho(f_n, f) = \|f_n - f\|_\infty \to 0$ with n. Thus $f_n \to f$ if and only if for all $\epsilon > 0$, there is a N such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$. That is, $f_n \to f$ in $C_b(X)$ or $\ell^\infty(X)$ if and only if $f_n \to f$ uniformly on X. For this reason the supremum norm $\|\cdot\|_\infty$ is often also called the uniform norm.

Subspace Magic

Proposition

Suppose that X is a metric space and that $f_n \in C(X)$ for all $n \in \mathbf{N}$. If (f_n) converges uniformly to a function $f : X \to \mathbf{C}$, then f is continuous.

Remark

Colloquially, the uniform limit of continuous functions is continuous.

Proof.

This is also part of the requirements for your "analyst license". I'm letting you show off for homework. □

Corollary

If X is a metric space, then $C_b(X)$ is complete in the uniform norm.

Proof.

The previous proposition implies that $C_b(X)$ is a closed subspace of the complete metric space $\ell^\infty(X)$. □

That's Enough for Day Two

- That is enough for now.