Math 73/103: Fall 2020
Lecture 20

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Getting Started

- We should be recording!
- Questions?
- Problems 36–45 will be due a week from Friday—Friday, November 6^{th}. We're taking Wednesday next week off in case anyone needs to recover from Tuesday.
- I made some minor alterations to the problems assigned for today. So if you are working ahead, you may want to check the assignments page again.
- I added some comments on the Cantor set, ternary expansions, and the Cantor-Lebesgue function on the assignment page. That material is purely for fun and not required. page.

Absolute Continuity

Definition

Suppose that μ and ν are two measures on $(X, \mathcal{M})$. (Remember that without an adjective, "measure" always means "positive measure".) We say that μ is absolutely continuous with respect to μ—written $\nu \ll \mu$—if $\mu(E) = 0$ implies that $\nu(E) = 0$.

Example (See Lecture 14)

Suppose that $f : (X, \mathcal{M}) \to [0, \infty]$ is measurable. Then

$$\nu(E) = \int_E f(x)\, d\mu(x) \quad \text{for } E \in \mathcal{M}$$

defines a measure ν on $(X, \mathcal{M})$ and $\nu \ll \mu$. Furthermore, if $g : X \to [0, \infty]$ is measurable, then

$$\int_X g(x)\, d\nu(x) = \int_X g(x) f(x)\, d\mu(x). \qquad (\ddagger)$$

return

A Bit Better

Lemma

Suppose that f, μ and ν are as in the previous example*. Then if $g \in \mathcal{L}^1(X, \mathcal{M}, \nu)$, we have $gf \in \mathcal{L}^1(X, \mathcal{M}, \mu)$ and*

$$\int_X g(x)\, d\nu(x) = \int_X g(x) f(x)\, d\mu(x).$$

Proof.

I'll leave the proof for a homework problem. □

σ-finite measures

Definition

We say that $(X, \mathcal{M}, \mu)$ is a σ-finite measure space or that μ is a σ-finite measure if we can find countably many sets $A_n \in \mathcal{M}$ such that $X = \bigcup_{n=1}^{\infty} A_n$ such that $\mu(A_n) < \infty$ for all n.

Remark

If $(X, \mathcal{M}, \mu)$ is σ-finite we can alternatively insist that the A_n above are either pairwise disjoint or nested with $A_n \subset A_{n+1}$. In the first case, "disjointify", in the second let $A_n' = A_1 \cup \cdots \cup A_n$.

Example

Lebesgue measure on $\mathbf{R}$ is a σ-finite measure (HW#37(a)). However, counting measure on $(\mathbf{R}, \mathcal{P}(\mathbf{R}))$ is not.

Radon-Nikodym Theorem

Theorem (Radon-Nikodym)

Suppose that μ and ν are σ-finite measures on $(X, \mathcal{M})$ such that $\nu \ll \mu$. Then there is a measurable function $f : X \to [0, \infty)$ such that

$$\nu(E) = \int_E f(x)\,d\mu(x) \quad \text{for all } E \in \mathcal{M}. \tag{‡}$$

If $g : X \to [0, \infty)$ is another such function satisfying (‡), then $f = g$ for μ-almost all x.

Proof.

Assume to begin with that $\mu(X) < \infty$ and $\nu(X) < \infty$. For all $c > 0$, let $\{\, P(c), N(c) \,\}$ be a Hahn Decomposition for $\nu - c\mu$. Now we "disjointify" $\bigcup_{k=1}^{\infty} N(kc)$.

Proof

Proof Continued.

That is,

$$A_1 = N(c), \quad \text{and for } k \geq 2,$$
$$A_k = N(kc) \setminus \bigcup_{j=1}^{k-1} N(jc) = N(kc) \cap \bigcap_{j=1}^{k-1} P(jc).$$

Thus if $E \subset A_k$ is measurable, then $E \subset N(kc)$ implies $\nu(E) - kc\mu(E) \leq 0$ and $E \subset P\big((k-1)c\big)$ implies $\nu(E) - (k-1)c\mu(E) \geq 0$. Therefore, $E \subset A_k$ implies

$$(k-1)c\mu(E) \leq \nu(E) \leq kc\mu(E). \qquad (*)$$

return

Proof

Proof Continued.

Now let

$$B = X \setminus \bigcup_{k=1}^{\infty} A_k = X \setminus \bigcup_{k=1}^{\infty} N(kc) = \Big(\bigcup_k N(kc)\Big)^C = \bigcap_{k=1}^{\infty} P(kc)$$

Since $B \subset P(kc)$,

$$0 \leq kc\mu(B) \leq \nu(B) \leq \nu(X) < \infty \quad \text{for all } k \in \mathbf{N}.$$

Thus $\mu(B) = 0$. Since $\nu \ll \mu$, we also have $\nu(B) = 0$.

Proof

Proof Continued.

Define

$$g_c(x) = \begin{cases} (k-1)c & \text{if } x \in A_k, \text{ and} \\ 0 & \text{if } x \in B. \end{cases}$$

Then g is well-defined on all of X. Since $\nu(B) = 0 = \mu(B)$ and in view of Equation (∗), we have

$$\int_E g_c(x)\,d\mu(x) \le \nu(E) \le \int_E (g_c + c)(x)\,d\mu(x) \le \int_E g_c(x)\,d\mu(x) + c\mu(X).$$

Thus, if we let $f_n = g_{2^{-n}}$, then for all $n, m \in \mathbf{N}$ we have

$$\int_E f_n(x)\,d\mu(x) \le \nu(E) \le \int_E f_m(x)\,d\mu(x) + 2^{-m}\mu(X). \qquad (\ddagger)$$

return

Proof

Proof Continued.

Since everything in sight is finite, we have for all $n \geq m \geq 1$ and $E \in \mathcal{M}$,

$$\Big| \int_E \big(f_n(x) - f_m(x)\big)\, d\mu(x) \Big| \leq 2^{-m}\mu(X).$$

Since this holds for $E^+ = \{\, x : f_n(x) - f_m(x) \geq 0 \,\}$ as well as $E^- = \{\, x : f_n(x) - f_m(x) \leq 0 \,\}$, we must have

$$\int_X |f_n(x) - f_m(x)|\, d\mu(x) \leq 2^{-m+1}\mu(X) \quad \text{when } n \geq m \geq 1.$$

Therefore $\{\, f_n \,\}$ is Cauchy in $L^1(\mu)$, and there is a $f \in \mathcal{L}^1(\mu)$ such that $f_n \to f$ in $L^1(\mu)$. □

Proof

Proof Continued.

Since $f_n \to f$ in $L^1(\mu)$, there is a subsequence $(f_{n_k}) \to f$ pointwise μ-almost everywhere. Thus we can assume that $f(x) \geq 0$ for all x (Why?). Since

$$\left| \int_E f_n \, d\mu - \int_E f \, d\mu \right| \leq \int_E |f_n - f| \, d\mu = \|f_n - f\|_1 \to 0,$$

we have $\lim_n \int_E f_n(x) \, d\mu(x) = \int_E f(x) \, d\mu(x)$. Now by ▸ Equation (‡) it follows that

$$\nu(E) = \lim_{n \to \infty} \int_E f_n(x) \, d\mu(x) = \int_E f(x) \, d\mu(x).$$

This completes the existence part of the proof when μ and ν are finite measures. The uniqueness statement is left for homework.

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

The General Case

Proof of the General Case.

Since μ and ν are both σ-finite, we can suppose that $X = \bigcup_{n=1}^{\infty} X_n$ with $\mu(X_n) < \infty$ and $\nu(X_n) < \infty$ and $X_n \subset X_{n+1}$. Applying the previous argument to $(X_n, \mathcal{M}(X_n))$ we have a function $h_n : X_n \to [0, \infty)$ such that

$$\nu(E) = \int_E h_n(x)\, d\mu(x) \quad \text{for all } E \in \mathcal{M}(X_n).$$

We can extend h_n to all of X be setting $h_n(x) = 0$ if $x \notin X_n$. If $n \leq m$ and $E \subset X_n$ is measurable, then

$$\int_E h_n(x)\, d\mu(x) = \nu(E) = \int_E h_m(x)\, d\mu(x).$$

Since $E \subset X_n$ is arbitrary, $h_n(x) = h_m(x)$ for μ-almost all $x \in X_n$.

Proof

Proof Continued.

Let

$$f_n(x) = \sup\{\, h_1(x), \dots, h_n(x)\,\}$$

Then $f_n \sim h_n$. Furthermore, $f_n \nearrow f$ for a measurable function $f : X \to [0, \infty]$. If $E \in \mathcal{M}$, then

$$\begin{aligned}\nu(E) &= \lim_{n\to\infty} \nu(E \cap X_n) = \lim_n \int_E h_n(x)\, d\mu(x) \\ &= \lim_n \int_E f_n(x)\, d\mu(x) \\ &\overset{\text{MCT}}{=} \int_E f(x)\, d\mu(x).\end{aligned}$$

One More Thing

Proof Continued.

Let $A = \{ x : f(x) = \infty \}$. Since

$$\int_E f(x)\,d\mu(x) = \int_E h_n(x)\,d\mu(x) \quad \text{for all } E \in \mathcal{M}(X_n),$$

it follows that $f(x) = h_n(x)$ for μ-almost all $x \in X_n$. Therefore $\mu(A \cap X_n) = 0$. Thus

$$\mu(A) = \lim_n \mu(A \cap X_n) = 0.$$

Therefore, we can choose f so that $f(X) \subset [0, \infty)$.

Uniqueness is a homework problem. □

Remark

In the preceding theorem, we call the function $f : X \to [0, \infty)$ such that

$$\nu(E) = \int_E f(x)\, d\mu(x)$$

"the" Radon-Nikodym derivative of ν with respect to μ. The notation $f = \frac{d\nu}{d\mu}$ is often employed. Thus, by our homework problem, for all $g \in \mathcal{L}^1(\nu)$, we have $g \frac{d\nu}{d\mu} \in \mathcal{L}^1(\mu)$ and

$$\int_X g\, d\nu = \int_X g \frac{d\nu}{d\mu}\, d\mu$$

which may at least explain the terminology and notation. We get away with saying "the" as $\frac{d\nu}{d\mu}$ is determined μ-almost everywhere.

Break Time

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Outer Measures Again

Definition

Recall that an algebra of subsets of a set X is a collection $\mathcal{A} \subset \mathcal{P}(X)$ containing X which is closed under complements and finite unions. A function $\rho : \mathcal{A} \to [0, \infty]$ is called a pre-measure on $\mathcal{A}$ if $\rho(\emptyset) = 0$ and whenever $\{ E_n \}_{n=1}^{\infty} \subset \mathcal{A}$ is a pairwise disjoint family such that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$, then

$$\rho\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \rho(E_n).$$

Remark

If the requirement that a pre-measure be countably additive on the algebra $\mathcal{A}$ seems a high bar, we can be comforted by the observation that algebras are much more tame creatures that σ-algebras. So unlike the case for measures, we will be able to build interesting pre-measures—well, at least one anyway.

Why Pre-Measures?

Proposition

Let $\mathcal{A}$ be an algebra of sets in X and $\rho : \mathcal{A} \to [0, \infty]$ a pre-measure on $\mathcal{A}$.

1. *The map $\rho^* : \mathcal{P}(X) \to [0, \infty]$ given by*

$$\rho^*(E) = \inf\Big\{ \sum_{k=1}^{\infty} \rho(A_k) : \text{each } A_k \in \mathcal{A} \text{ and } E \subset \bigcup_k A_k \Big\}$$

   is an outer measure on X.

2. *$\rho^*(A) = \rho(A)$ for all $A \in \mathcal{A}$.*
3. *Every $A \in \mathcal{A}$ is ρ^*-measurable.*

Proof.

We will leave this as a homework exercise. □

Measures from Pre-Measures

Theorem

Suppose that ρ is a pre-measure on an algebra $\mathcal{A}$ of sets in X. Then there is a measure μ on the σ-algebra $\mathcal{M} := \mathcal{M}(\mathcal{A})$ generated by $\mathcal{A}$ such that $\mu(E) = \rho^*(E)$ for all $E \in \mathcal{M}$. In particular, $\mu(A) = \rho(A)$ for all $A \in \mathcal{A}$. If ν is any other measure on $\mathcal{M}$ extending ρ on $\mathcal{A}$, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$ with equality if $\mu(E) < \infty$. If ρ is σ-finite, then μ is the unique extension of ρ to $\mathcal{M}$.

Proof.

We already know that ρ^* restricts to a measure on the ρ^*-measurable sets $\mathcal{M}^*$. Since you will prove that $\mathcal{A} \subset \mathcal{M}^*$, we have $\mathcal{M} := \mathcal{M}(\mathcal{A}) \subset \mathcal{M}^*$. So we just let $\mu = \rho^*|_{\mathcal{M}}$. This gives us an extension μ as claimed.

Proof

Proof of Uniqueness.

Suppose ν is a measure on $\mathcal{M}$ extending ρ. If $E \in \mathcal{M}$ and $E \subset \bigcup_k A_k$ with $A_k \in \mathcal{A}$, then

$$\nu(E) \leq \sum_k \nu(A_k) = \sum_k \rho(A_k).$$

Therefore $\nu(E) \leq \rho^*(E) = \mu(E)$. Also, if $A = \bigcup_k A_k$, then

$$\nu(A) = \lim_n \nu\Big(\bigcup_{k=1}^n A_k\Big) = \lim_n \mu\Big(\bigcup_{k=1}^n A_k\Big) = \mu(A).$$

If $\mu(E) < \infty$ and $\epsilon > 0$, then we can choose that A_k's so that $\mu(A) \leq \sum_k \mu(A_k) = \sum_k \rho(A_k) < \rho^*(E) + \epsilon = \mu(E) + \epsilon$. Hence $\mu(A \setminus E) < \epsilon$. Then

$$\begin{aligned} \mu(E) \leq \mu(A) = \nu(A) &= \nu(E) + \nu(A \setminus E) \\ &\leq \nu(E) + \mu(A \setminus E) \leq \nu(E) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\mu(E) \leq \nu(E)$. Hence $\mu(E) = \nu(E)$.

Proof

Proof Continued.

In the general σ-finite case, suppose that $X = \bigcup A_k$ with $A_k \in \mathcal{A}$ and $\rho(A_k) < \infty$. We can assume that the A_k are pairwise disjoint. Then for any $E \in \mathcal{M}$,

$$\mu(E) = \sum_k \mu(E \cap A_k) = \sum_k \nu(E \cap A_k) = \nu(E).$$

Thus $\nu = \mu$ in the σ-finite case. □

That's Enough for Today

- That is enough for now.