Math 73/103: Fall 2020
Lecture 21

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Getting Started

- We should be recording!
- Questions?
- Problems 36–45 will be due Friday, November 6^{th} via `gradescope`.
- There is no lecture Wednesday next week.
- I added some comments on the Cantor set, ternary expansions, and the Cantor-Lebesgue function on the assignment page. That material is purely for fun and not required. page.

Product Measures

- Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. We want to build a measure $\mu \times \nu$ on the Cartesian product $X \times Y$.
- If $A \in \mathcal{M}$ and $B \in \mathcal{N}$, then we call $A \times B$ a measurable rectangle.
- Naturally, we want $\mu \times \nu(A \times B) = \mu(A)\nu(B)$.
- We let $\mathcal{R} = \{ A \times B : A \in \mathcal{M} \text{ and } B \in \mathcal{N} \}$ be the set of all measurable rectangles in $X \times Y$.
- We will define $\mathcal{M} \otimes \mathcal{N}$ be the σ-algebra in $X \times Y$ generated by $\mathcal{R}$.
- Note that $\mathcal{R}$ is closed under intersection.
- If $A \times B \in \mathcal{R}$, then $(A \times B)^C = (A^C \times Y) \cup (X \times B^C)$ which is a disjoint union of measurable rectangles.

The Algebra of Measurable Rectangles

Lemma

Let $\mathcal{A}$ be the collection of finite unions of disjoint measurable rectangles. Then $\mathcal{A}$ is an algebra in $X \times Y$.

Proof.

Suppose that $E, F \in \mathcal{R}$. Then as above, $F^C = R_1 \cup R_2$ with $R_k \in \mathcal{R}$ and $R_1 \cap R_2 = \emptyset$. Then $E \setminus F = E \cap F^C = (E \cap R_1) \cup (E \cap R_2) \in \mathcal{A}$. Then $E \cup F = E \setminus F \cup F \in \mathcal{A}$.

Now suppose $E_1, \dots, E_n \in \mathcal{R}$. I claim $E_1 \cup \cdots \cup E_n \in \mathcal{A}$. Since we have the case $n = 2$, proceed by induction. Assume $E_1 \cup \cdots \cup E_{n-1} \in \mathcal{A}$. Then $E_1 \cup \cdots \cup E_{n-1} = \bigcup_{k=1}^m F_k$ with each $F_k \in \mathcal{R}$ and $F_i \cap F_j = \emptyset$ if $i \neq k$.

Proof Continued.

Now

$$E_1 \cup \cdots \cup E_n = E_n \cup \bigcup_{k=1}^{m} F_k \setminus E_n \in \mathcal{A}.$$

This proves the claim, and it easily follows that $\mathcal{A}$ is closed under unions.

But if $E = \bigcup_{k=1}^{n} R_k \in \mathcal{A}$, then

$$\begin{aligned} E^C &= \bigcap_{k=1}^{n} R_k^C = \bigcap_{k=1}^{n} R_k^1 \cup R_k^2 \\ &= \bigcup \{\, R_1^{k_1} \cap R_2^{k_2} \cap \cdots \cap R_n^{k_n} : \text{where } k_j \text{ equals 1 or 2} \,\} \\ &\in \mathcal{A}. \end{aligned}$$

Thus $\mathcal{A}$ is an algebra as claimed. □

Getting to a Pre-Measure

Lemma

Suppose that $E = A \times B \in \mathcal{R}$ *and that*

$$E = \bigcup_{k=1}^{\infty} A_k \times B_k$$

where $A_k \times B_k \in \mathcal{R}$ *are pairwise disjoint. Then*

$$\mu(A)\nu(B) = \sum_{k=1}^{\infty} \mu(A_k)\mu(B_k).$$

Proof

Proof.

If $(x, y) \in X \times Y$, then

$$\mathbb{1}_A(x)\mathbb{1}_B(y) = \mathbb{1}_{A\times B}(x, y) = \sum_{k=1}^{\infty} \mathbb{1}_{A_k\times B_k}(x, y) = \sum_{k=1}^{\infty} \mathbb{1}_{A_k}(x)\mathbb{1}_{B_k}(y).$$

Now hold y fixed and integrate w.r.t. x:

$$\mu(A)\mathbb{1}_B(y) = \sum_{k=1}^{\infty} \mu(A_k)\mathbb{1}_{B_k}(y).$$

Now integrate w.r.t. y:

$$\mu(A)\nu(B) = \sum_{k=1}^{\infty} \mu(A_k)\nu(B_k). \quad \square$$

Our Pre-Measure

Proposition

There is a unique pre-measure π on $\mathcal{A}$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for all $A \times B \in \mathcal{R}$.

Proof.

Suppose $\{ A_i \times B_i \}_{i=1}^{n}$ and $\{ C_j \times D_j \}_{j=1}^{m}$ are elements of $\mathcal{A}$ such that

$$\bigcup_i A_i \times B_i = \bigcup_j C_j \times D_j.$$

Then

$$A_i \times B_i = \bigcup_{j=1}^{m} A_i \cap C_j \times B_i \cap D_j$$

$$C_j \times D_j = \bigcup_{i=1}^{n} A_i \cap C_j \times B_i \cap D_j$$

are both disjoint unions.

Proof

Proof Continued.

Now we an use our lemma to conclude that

$$\sum_{i=1}^{n} \mu(A_i)\nu(B_i) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mu(A_i \cap C_j)\nu(B_i \cap D_j) = \sum_{j=1}^{m} \mu(C_j)\nu(D_j).$$

Therefore we get a well-defined function $\pi : \mathcal{A} \to [0, \infty]$ such that

$$\pi\Big(\bigcup_{k=1}^{n} A_k \times B_k\Big) = \sum_{k=1}^{n} \mu(A_k)\nu(B_k).$$

Proof

Proof Continued.

Now suppose that $E = \bigcup_{j=1}^{n} C_j \times D_j \in \mathcal{A}$ and that

$$E = \bigcup_{k=1}^{\infty} A_k \times B_k$$

is the pairwise disjoint union of measurable rectangles. Then we can use our lemma to see that

$$\begin{aligned}
\pi(E) = \sum_{j=1}^{n} \mu(C_j)\nu(D_j) &= \sum_{j=1}^{n} \sum_{k=1}^{\infty} \mu(C_j \cap A_k)\nu(D_j \cap B_k) \\
&= \sum_{k=1}^{\infty} \mu(A_k)\nu(B_k) \\
&= \sum_{k=1}^{\infty} \pi(A_k \times B_k).
\end{aligned}$$

Proof

Proof Continued.

It is not hard to use this to show that π is a pre-measure on $\mathcal{A}$: if E is the disjoint union $\bigcup_{k=1}^{\infty} E_k$ with $E_k = \bigcup_{j=1}^{n_k} R_j^k \in \mathcal{A}$, then

$$E = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{n_k} R_j^k$$

is a countable pairwise disjoint union of rectangles. Thus

$$\pi(E) = \sum_{k=1}^{\infty} \sum_{j=1}^{n_k} \pi(R_j^k) = \sum_{k=1}^{\infty} \pi(E_k).$$

Uniqueness is straightforward. □

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Payoff

Definition

If $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces, then the product measure $\nu \times \nu$ or simply the product of μ and ν is the measure on $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ coming from the pre-measure π defined above.

Remark (Uniqueness)

If μ and ν are σ-finite, then so is the pre-measure π. Then $\mu \times \nu$ is also σ-finite. Hence $\mu \times \nu$ is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ such that

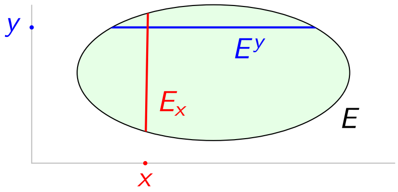
$$\mu \times \nu(A \times B) = \mu(A)\nu(B) \quad \text{for all } A \times B \in \mathcal{R}.$$

Definition

If $E \subset X \times Y$ and $(x, y) \in X \times Y$. Then

$$E_x = \{\, y \in Y : (x, y) \in E \,\} \quad \text{and} \quad E^y = \{\, x \in X : (x, y) \in E \,\}.$$

If $f : X \times Y \to Z$ is a function then $f_x : Y \to Z$ is given by $f_x(y) = f(x, y)$ and $f^y : X \to Z$ is given by $f^y(x) = f(x, y)$.



Example

$$(\mathbb{1}_E)_x = \mathbb{1}_{E_x}$$
$$(\mathbb{1}_E)^y = \mathbb{1}_{E^y}$$

Measurable Sections

Proposition

Suppose that $E \in \mathcal{M} \otimes \mathcal{N}$. *Then for all* $(x, y) \in X \times Y$, $E_x \in \mathcal{N}$ *and* $E^y \in \mathcal{M}$. *If* $f : X \times Y \to \mathbf{C}$ *is* $\mathcal{M} \otimes \mathcal{N}$*-measurable, then* $f_x : Y \to \mathbf{C}$ *is* $\mathcal{N}$*-measurable and* $f^y : X \to \mathbf{C}$ *is* $\mathcal{M}$*-measurable.*

Proof.

Let $\mathcal{P} = \{\, E \subset X \times Y : E_x \in \mathcal{N} \text{ and } E^y \in \mathcal{M} \,\}$. If $A \times B \in \mathcal{R}$, then

$$(A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \quad \text{and} \quad (A \times B)^y = \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{if } y \notin B. \end{cases}$$

Therefore $\mathcal{R} \subset \mathcal{P}$. Since it is not hard to check that $\mathcal{P}$ is a σ-algebra, we have $\mathcal{M} \otimes \mathcal{N} \subset \mathcal{P}$. This proves the first assertion.

Proof

Proof Continued.

For the second assertion, convince yourself that

$$(f_x)^{-1}(V) = \big(f^{-1}(V)\big)_x \quad \text{and} \quad (f^y)^{-1}(V) = \big(f^{-1}(V)\big)^y.$$

Therefore the second assertion follows from the second. □

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Monotone Classes

Definition

A subset $\mathscr{C} \subset \mathcal{P}(X)$ is called a monotone class if is closed under increasing countable unions and decreasing countable intersections.

Example

Every σ-algebra is a monotone class. The collection $\mathscr{C}$ of intervals in $\mathbf{R}$ (including the empty set and points) is a monotone class that is not a σ-algebra.

Lemma

Given any subset $\mathcal{E} \subset \mathcal{P}(X)$, there is a smallest monotone class $\mathscr{C}(\mathcal{E})$ containing $\mathcal{E}$. We call $\mathscr{C}(\mathcal{E})$ the monotone class generated by $\mathcal{E}$.

Proof.

The intersection of monotone classes is a monotone class. □

The Monotone Class Lemma

Theorem (The Monotone Class Lemma)

Suppose that $\mathcal{A}$ is an algebra of sets in X. Then the monotone class $\mathscr{C}(\mathcal{A})$ generated by $\mathcal{A}$ coincides with the σ-algebra $\mathcal{M}(\mathcal{A})$ generated by $\mathcal{A}$. In particular, $\mathscr{C}(\mathcal{A})$ is a σ-algebra.

Proof.

Since $\mathcal{M}(\mathcal{A})$ is a monotone class containing $\mathcal{A}$, $\mathscr{C}(\mathcal{A}) \subset \mathcal{M}(\mathcal{A})$. If $E \in \mathscr{C}(\mathcal{A})$, let

$$\mathcal{D}(E) = \{\, F \in \mathscr{C}(\mathcal{A}) : E \setminus F,\ F \setminus E, \text{ and } F \cap E \text{ are all in } \mathscr{C}(\mathcal{A}) \,\}$$

Check that

1. $\emptyset, X \in \mathcal{D}(E)$,
2. $F \in \mathcal{D}(E)$ implies $E \in \mathcal{D}(F)$, and
3. $\mathcal{D}(E)$ is a monotone class.

Proof

Proof Continued.

Since $\mathcal{A}$ is an algebra, $\mathcal{A} \subset \mathcal{D}(E)$ for all $E \in \mathcal{A}$. Since $\mathcal{D}(E)$ is a monotone class, $\mathscr{C}(\mathcal{A}) \subset \mathcal{D}(E)$ for all $E \in \mathcal{A}$. Thus $\mathscr{C}(\mathcal{A}) = \mathcal{D}(E)$ for all $E \in \mathcal{A}$.

Therefore $F \in \mathcal{D}(E)$ whenever $E \in \mathcal{A}$ and $F \in \mathscr{C}(\mathcal{A})$. Then using (2) above,

$E \in \mathcal{D}(F)$ for all $F \in \mathscr{C}(\mathcal{A})$ and $E \in \mathcal{A}$. Then since $\mathcal{D}(F)$ is a monotone class,

$\mathscr{C}(\mathcal{A}) \subset \mathcal{D}(F)$ for all $F \in \mathscr{C}(\mathcal{A})$. Thus $\mathcal{D}(F) = \mathscr{C}(\mathcal{A})$ for all $F \in \mathscr{C}(\mathcal{A})$.

Therefore if $E, F \in \mathscr{C}(\mathcal{A})$, then $E \cap F \in \mathscr{C}(\mathcal{A})$ and $E \setminus F \in \mathscr{C}(\mathcal{A})$. Since $X, \emptyset \in \mathscr{C}(\mathcal{A})$, it follows that $\mathscr{C}(\mathcal{A})$ is an algebra. Now if $\{ E_k \}_{k=1}^{\infty} \subset \mathscr{C}(\mathcal{A})$, then for each n, $\bigcup_{k=1}^{n} E_k \in \mathscr{C}(\mathcal{A})$. But then $\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} E_k = \bigcup_{k=1}^{\infty} E_k \in \mathscr{C}(\mathcal{A})$. Therefore $\mathscr{C}(\mathcal{A})$ is a σ-algebra containing $\mathcal{A}$. Then $\mathcal{M}(\mathcal{A}) \subset \mathscr{C}(\mathcal{A})$. □

That's Enough for Today

- That is enough for now.