Math 73/103: Fall 2020
Lecture 22

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- We should be recording!
- Questions?
- Problems 36–45 will be due Friday, November 6^{th} via `gradescope`.
- There is no lecture Wednesday.

Last Time

- Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces.
- As you will see, we will soon want to require μ and ν to be σ-finite.
- We let $\mathcal{M} \otimes \mathcal{N}$ be the σ-algebra in $X \times Y$ generated by the measurable rectangles.
- Then we get a pre-measure π on the algebra of finite unions of disjoint measurable rectangles where $\pi(A \times B) = \mu(A)\nu(B)$.
- Then the product measure $\mu \times \nu$ is the measure on $\mathcal{M} \otimes \mathcal{N}$ coming from the outer measure associated to π.
- If μ and ν are σ-finite, then $\mu \times \nu$ is the unique measure on $(X \times Y, \mathcal{M} \otimes \mathcal{N})$ such that $\mu \times \nu(A \times B) = \mu(A)\nu(B)$.

Proposition

Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are both σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$, then

$$x \mapsto \nu(E_x) \quad \text{and} \quad y \mapsto \mu(E^y) \tag{‡}$$

are measurable functions from $X \to [0, \infty]$ and $Y \to [0, \infty]$, respectively. Furthermore,

$$\mu\times\nu(E) = \int_X \nu(E_x)\,d\mu(x) = \int_Y \mu(E^y)\,d\nu(y).$$

▸ return

Remark

Note that it is necessary to consider extended nonnegative valued functions in (‡)! Furthermore, we allow the value ∞ for $\mu\times\nu(E)$.

Proof

Proof.

To start, suppose that μ and ν are both finite measures. Let $\mathscr{C}$ be the collection of $E \in \mathcal{M} \otimes \mathcal{N}$ such that the assertions in the proposition hold. If $A \times B \in \mathcal{R}$, then

$$\nu(E_x) = \nu(B)\mathbb{1}_A(x) \quad \text{and} \quad \mu(E^y) = \mu(A)\mathbb{1}_B(y).$$

Clearly these are measurable functions and the integral formula holds. Therefore $\mathcal{R} \subset \mathscr{C}$. Since $\mathcal{A}$ is the collection of finite disjoint unions from $\mathcal{R}$, we also have $\mathcal{A} \subset \mathscr{C}$.

By the Monotone Class Lemma, it will suffice—in the finite case—to see that $\mathscr{C}$ is a monotone class: then $\mathscr{C} \overset{\text{MCL}}{=} \mathcal{M}(\mathcal{A}) := \mathcal{M} \otimes \mathcal{N}$.

Proof

Proof Continued.

Suppose $\{ E_n \} \subset \mathscr{C}$ with $E_n \subset E_{n+1}$. Let $E = \bigcup_{n=1}^{\infty} E_n$, $f_n(y) = \mu(E_n^y)$, and $f(y) = \mu(E^y)$. By assumption each f_n is measurable, and $f_n \nearrow f$. Hence f is measurable and

$$\begin{aligned} \int_Y \mu(E^y)\,d\nu(y) &= \int_Y f(y)\,d\mu(y) \overset{\text{MCT}}{=} \lim_n \int_Y f_n(y)\,d\nu(y) \\ &= \lim_n \int_Y \mu(E_n^y)\,d\nu(y) = \lim_n \mu\times\nu(E_n) \\ &= \mu\times\nu(E). \end{aligned}$$

Since a similar argument implies $\int_X \nu(E_x)\,d\mu(x) = \mu\times\nu(E)$, we have $E \in \mathscr{C}$.

Proof

Proof Continued.

Now we suppose that $E = \bigcap E_n$ with $E_{n+1} \subset E_n$. We let $f_n(y) = \mu(E_n^y)$, and $f(y) = \mu(E^y)$ as before. Now $f_n \searrow f$, and as before, f is measurable. Since $\mu\times\nu$ is finite,

$$\int_Y f_1(y)\,d\mu = \int_Y \mu(E_1^y)\,d\nu(y) = \mu\times\nu(E_1) < \infty.$$

Hence $f_1 \in \mathcal{L}^1(\mu)$. Therefore the LDCT implies that

$$\lim_n \int_Y f_n(y)\,d\nu(y) = \int_Y f(y)\,d\nu(y),$$

and as on the previous slide $\int_Y \mu(E^y)\,d\nu(y) = \mu\times\nu(E)$. Similarly, $\int_X \nu(E_x)\,d\mu(x) = \mu\times\nu(E)$ and $E \in \mathscr{C}$. Therefore $\mathscr{C}$ is a monotone class containing $\mathcal{A}$ and $\mathscr{C} = \mathcal{M}\otimes\mathcal{N}$.

This takes care of the finite case.

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Proof in General.

Now we suppose $X = \bigcup X_k$ and $Y = \bigcup Y_k$ with $\mu(X_k) < \infty$, $\nu(Y_k) < \infty$, $X_k \subset X_{k+1}$, and $Y_{k+1} \subset Y_k$ for all k. Let

$$\mu_k(E) = \mu(E \cap X_k) \quad \text{and} \quad \nu_k(F) = \nu(F \cap Y_k).$$

Then μ_k and ν_k are finite measures. If $f \geq 0$, then I claim that

$$\int_X f(x)\, d\mu_k(x) = \int_X \mathbb{1}_{X_k}(x) f(x)\, d\mu(x).$$

To see this, notice that it is clearly true if f is a MNNSF. Now use the MCT.

Proof

Proof Continued.

We also let

$$(\mu\times\nu)_k(G) = \mu\times\nu\big(G \cap (X_k \times Y_k)\big).$$

Notice that if $G = A \times B \in \mathcal{R}$, then

$$\begin{aligned}(\mu\times\nu)_k(A \times B) &= \mu\times\nu\big((A \cap X_k) \times (B \cap Y_k)\big)\\ &= \mu(A \cap X_k)\nu(B \cap Y_k)\\ &= \mu_k(A)\nu_k(B)\end{aligned}$$

Since μ and ν are finite, we get

$$(\mu\times\nu)_k = \mu_k \times \nu_k$$

by uniqueness.

Proof

Proof Continued.

Now

$$\mu\times\nu\big(E\cap(X_k\times Y_k)\big) = (\mu\times\nu)_k(E) = \mu_k\times\nu_k(E)$$
$$= \int_X \nu_k(E_x)\,d\mu_k(x)$$
$$= \int_X \mathbb{1}_{X_k}(x)\nu\big(E_x\cap Y_k\big)\,d\mu(x).$$

Therefore

$$\mu\times\nu(E) = \lim_k \mu\times\nu\big(E\cap(X_k\times Y_k)\big)$$
$$= \lim_k \int_X \mathbb{1}_{X_k}(x)\nu\big(E_x\cap Y_k\big)\,d\mu(x)$$
$$\overset{\text{MCT}}{=} \int_X \nu(E_x)\,d\mu(x).$$

Since a similar argument works for the Y-integral, we are done. □

Non-Negative Functions

Definition

If $(X, \mathcal{M}, \mu)$ is a measure space, we will write $L^+(X, \mathcal{M}, \mu)$ for collection of measurable functions $f : X \to [0, \infty]$.

Remark

We allow the value ∞ only to cope with situations were we can't avoid it. Note that if $f \in L^+(X, \mathcal{M}, \mu)$ is such that

$$\int_X f(x)\,d\mu(x) < \infty,$$

then $N = \{\, x : |f(x)| = \infty \,\}$ is a null set, and $\mathbb{1}_{X \setminus N} \cdot f$ maps into $[0, \infty) \subset \mathbf{C}$ and $\mathbb{1}_{X \setminus N} \cdot f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$. Thus there is no real harm in thinking of such a function as defining a class $[f] \in L^1(X, \mathcal{M}, \mu)$.

Theorem (Tonelli)

Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite and that $f \in L^+(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$.

1. *For all $(x, y) \in X \times Y$, $f_x \in L^+(Y)$ and $f^y \in L^+(X)$.*
2. *If*
$$g(x) := \int_Y f(x,y)\,d\nu(y) \quad \text{and} \quad h(y) = \int_X f(x,y)\,d\mu(x),$$
then $g \in L^+(X)$ and $h \in L^+(Y)$.
3. *Moreover,*
$$\int_{X\times Y} f(x,y)\,d\mu\times\nu(x,y) = \int_X \int_Y f(x,y)\,d\nu(y)\,d\mu(x)$$
$$= \int_Y \int_X f(x,y)\,d\mu(x)\,d\nu(y).$$

Pedantry

Remark

I was devastated to learn that Winston Churchill probably never said "This is the type of arrant pedantry up with which I will not put" when accused of ending a sentence with a preposition. Nevertheless, some pedantry has its place. In item (2), I should probably have writen

$$g(x) = \int_Y f_x(y)\,d\mu(y) \quad \text{and} \quad h(y) = \int_X f^y(x)\,d\mu(x).$$

Then item (3) is more properly written

$$\int_{X\times Y} f(x,y)\,d\mu\times\nu(x,y) = \int_X g(x)\,d\mu(x) = \int_Y h(y)\,d\nu(y).$$

But I believe the given statement is more intuitive as written. In particular, it reminds of us the good ol' days of Calculus III.

Proof

Proof.

If $f = \mathbb{1}_E$, then the result is just a restatement of the previous proposition. Therefore the result holds for MNNSFs. If $f \in L^+(X \times Y)$, then there are MNNSFs $f_n \nearrow f$. Then $(f_n)_x \nearrow f_x$ and $(f_n)^y \nearrow f^y$. Thus both are measurable and item (1) holds. If $g_n(x) = \int_Y f_n(x,y)\,d\nu(y)$ and $h_n(y) = \int_X f_n(x,y)\,d\mu(x)$, then the MCT implies that $g_n \nearrow g$ and $h_n \nearrow h$. Thus item (2) is valid. Again by the MCT,

$$\begin{aligned}\int_X g(x)\,d\mu(x) &= \lim_n \int_X g_n(x)\,d\mu(x) \\ &= \lim_n \int_{X\times Y} f_n(x,y)\,d\mu{\times}\nu(x,y) \\ &\overset{\text{MCT}}{=} \int_{X\times Y} f(x,y)\,d\mu{\times}\nu(x,y).\end{aligned}$$

Similarly, $\int_Y h(y)\,d\nu(y) = \int_{X\times Y} f(x,y)\,d\mu{\times}\nu(x,y)$. This completes the proof. $\square$

Break Time

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The Fubini Theorem

Theorem (Fubini)

Suppose that $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *are* σ*-finite and that* $f \in \mathcal{L}^1(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \otimes \nu)$.

1. *For all* $(x, y) \in X \times Y$, *both* f_x *and* f^y *are measurable.*
2. *For* μ*-almost all* x, $f_x \in \mathcal{L}^1(Y)$ *and for* ν*-almost all* y, $f^y \in \mathcal{L}^1(X)$. *Moreover,*

$$g(x) = \begin{cases} \int_Y f(x,y)\,d\nu(y) & \text{if } f_x \in \mathcal{L}^1(Y), \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

   is in $\mathcal{L}^1(X)$ *while*

$$h(y) = \begin{cases} \int_X f(x,y)\,d\mu(x) & \text{if } f^y \in \mathcal{L}^1(X), \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

   is in $\mathcal{L}^1(Y)$.
3. *We have*

$$\int_{X \times Y} f(x,y)\,d\mu{\times}\nu(x,y) = \int_X g(x)\,d\mu(x) = \int_Y h(y)\,d\nu(y).$$

Proof.

Let $f = u + iv$. Then $\{ u^{\pm}, v^{\pm} \} \in \mathcal{L}^1(\mu\times\nu) \cap L^+(\mu \times \nu)$. Since the union of four null sets is a null set, we can reduce to the case that $f \in \mathcal{L}^1(\mu\times\nu) \cap L^+(\mu \times \nu)$. Then by part (1) of Tonelli's Theorem, f_x and f^y are measurable. This proves item (1).

Also by Tonelli,

$$\tilde{g}(x) = \int_Y f(x,y)\, d\nu(y)$$

is measurable (into $[0,\infty]$) and

$$\int_X \tilde{g}(x)\, d\mu(x) = \int_{X\times Y} f(x,y)\, d\mu\times\nu(x,y) < \infty.$$

Proof

Proof Continued.

Therefore $N = \{ x : \tilde{g}(x) = \infty \}$ is μ-null, and $g = \mathbb{1}_{X \setminus N} \cdot \tilde{g}$ is measurable. Since $g = \tilde{g}$ for μ-almost all x, $g \in \mathcal{L}^1(X)$. Furthermore,

$$\int_X g(x)\, d\mu(x) = \int_{X\times Y} f(x,y)\, d\mu\times\nu(x,y).$$

By symmetry, $h \in \mathcal{L}^1(Y)$ and

$$\int_Y h(y)\, d\nu(y) = \int_{X\times Y} f(x,y)\, d\mu\times\nu(x,y).$$

This completes the proof. □

Remark

In practice, you may be presented with an iterated integral

$$\int_X \int_Y f(x,y)\,d\nu(y)\,d\mu(x).$$

Even though the product measure is not on your radar, you want to interchange the order of integration. First, you must verify that f *is* $\mathcal{M}\otimes\mathcal{N}$*-measurable. This can be quite hard sometimes. Then you apply Tonelli to compute*

$$\int_X \int_Y |f(x,y)|\,d\nu(y)\,d\mu(x) = \int_{X\times Y} |f(x,y)|\,d\mu\times\nu(x,y)$$

to show that $f \in \mathcal{L}^1(\mu\times\nu)$*. Then you can apply Fubini to interchange the order of integration.*

Remark

Every Ph.D. program tells the story of the student who was puzzled by the request to evaluate

$$S = \int_0^\infty e^{-x^2}\, dx.$$

The story goes that after some minutes of silence one of the "kinder" examiners suggests "use Fubini".

The point of the joke is that you can't make sense of the hint unless you already know how to do the problem.

First

$$S^2 = \Big(\int_0^\infty e^{-x^2}\, dx\Big)\Big(\int_0^\infty e^{-x^2}\, dx\Big) = \int_0^\infty \int_0^\infty e^{-x^2-y^2}\, dx\, dy$$

Now Fubini

Now by Fubini (and Tonelli),

$$S^2 = \iint_Q e^{-x^2-y^2}\, dA$$

where $Q = \{\, (x,y); x,y \geq 0 \,\}$ and dA is the product measure $m \times m$ on $\mathbf{R}^2$. Then if $Q_N = \{\, (x,y) : x,y \geq 0 \text{ and } x^2+y^2 \leq N \,\}$,

$$S^2 = \lim_{N\to\infty} \iint_{Q_N} e^{-x^2-y^2}\, dA$$

which, but Fubini Again as well as a dose of polar coordinates, is

$$= \lim_{N\to\infty} \int_0^{\frac{\pi}{2}} \int_0^N e^{-r^2} r\, dr\, d\theta = \lim_{N\to\infty} \frac{\pi}{4}(1-e^{-N^2}) = \frac{\pi}{4}.$$

Therefore $S = \dfrac{\sqrt{\pi}}{2}$.

That's Enough for Today

- That is enough for now.