Math 73/103: Fall 2020
Lecture 24

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Getting Started

- We should be recording!
- Questions?
- Problems 36–45 are due today via `gradescope`.
- There is no Lecture 23.

Complete Product Measures

Remark

Even if $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *are complete measure spaces, it need not be the case that* $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ *is complete. As we shall see, working with* $\mathcal{M} \otimes \mathcal{N}$ *has many advantages, but there is also a natural prejudice for complete measures. So now we want to investigate the completion* $(X \times Y, \mathcal{L}, \lambda)$ *of* $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ *when both* μ *and* ν *are* σ*-finite.*

Homework

Lemma (Homework Problem #47)

Let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ where μ and ν are complete σ-finite measures.

1. *If $E \in \mathcal{M} \otimes \mathcal{N}$ and $\mu \times \nu(E) = 0$, then $\mu(E^y) = 0 = \nu(E_x)$ for μ-almost all x and ν-almost all y.*
2. *If f is $\mathcal{L}$-measurable and $f(x, y) = 0$ for λ-almost all (x, y), then there is a μ-null set M and a ν-null set M such that for all $x \notin M$ and $y \notin N$, f_x and f^y are integrable. Furthermore*

$$\int_X f^y(x)\, d\mu(x) = 0 = \int_Y f_x(y)\, d\nu(y).$$

return

Tonelli for the Completion

Theorem (Tonelli)

Let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ where μ and ν are complete *σ-finite* *measures. Suppose that $f \in L^+(X \times Y, \mathcal{L}, \lambda)$. Then there are null sets $M \subset X$ and $N \subset Y$ such that the following hold.*

1. *f_x and f^y are measurable if $x \notin M$ and $y \notin N$.*
2. *If $g(x) = \int_Y f(x, y)\, d\nu(y)$ if $x \notin M$ and 0 otherwise and $h(y) = \int_X f(x, y)\, d\mu(x)$ if $y \notin N$ and 0 otherwise, then $g \in L^+(X)$ and $h \in L^+(Y)$. Furthermore*
3. $$\int_{X \times Y} f\, d\lambda = \int_X g\, d\mu = \int_y h\, d\nu.$$

Fubini for the Completion

Theorem (Fubini)

Let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ where μ and ν are complete σ-finite measures. Suppose that $f \in \mathcal{L}^1(\lambda)$. Then there are null sets $M \subset X$ and $N \subset Y$ such that

4. *$f_x \in \mathcal{L}^1(\nu)$ if $x \notin M$ and $f^y \in \mathcal{L}^1(\mu)$ if $y \notin N$,*
5. *if we define $g(x) = \int_Y f(x, y)\, d\nu(y)$ when $x \notin M$ and 0 otherwise, and similarly for h, then $g \in \mathcal{L}^1(\mu)$ and $h \in \mathcal{L}^1(\nu)$. Furthermore*
6. $$\int_{X \times Y} f \, d\lambda = \int_X g \, d\mu = \int_y h \, d\nu.$$

Proof of Tonelli and Fubini

Proof.

Suppose that $f \in L^+(\lambda)$. Then by HW#39, there is a $h \in L^+(\mu \times \nu)$ such that $h = f$ λ-almost everywhere. Then $f = h + (f - h)$ and we can apply part (2) of our HW Lemma to $f - h$. Since $f_x = h_x + (f - h)_x$ and h_x is always measurable, f_x is measurable almost everywhere. By symmetry, so is f^y. This proves part (1). If f is also integrable, then h is integrable and h_x is integrable almost everywhere as is $(f - h)_x$ (by HW#39). Thus f_x (and by symmetry f^y) is integrable almost everywhere. Now part (4) follows by decomposing $f \in \mathcal{L}^1(\lambda)$ into a linear combination of positive functions.

Proof Continued.

If $f \in L^+(\lambda)$, then $x \mapsto g(x)$—essentially $x \mapsto \int_Y f_x \, d\nu$—is equal almost everywhere to $x \mapsto \int_Y h_x \, d\nu$, so g is measurable since μ is complete. By symmetry, we have established part (2). If f is also integrable, then h is integrable. Therefore $x \mapsto \int_Y h_x \, d\nu$ is integrable which implies g is. By symmetry, we have established part (5).

Parts (3) and (6) follow easily as the integrals all are given by their h-counterparts. □

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Definition

Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p < \infty$. Let $\mathcal{L}^p(X, \mathcal{M}, \mu)$ be the set of measurable functions $f : X \to \mathbf{C}$ such that

$$\int_X |f(x)|^p \, d\mu(x) < \infty.$$

If $f \in \mathcal{L}^p(X, \mathcal{M}, \mu)$, then we define its p-norm to be

$$\|f\|_p = \Big(\int_X |f(x)|^p \, d\mu(x)\Big)^{\frac{1}{p}}.$$

We let $L^p(X, \mathcal{M}, \mu)$ be the set of equivalence class in $\mathcal{L}^p(X, \mathcal{M}, \mu)$ where $f \sim g$ if $f(x) = g(x)$ for μ-almost all x. We let $\|[f]\|_p = \|f\|_p$.

Example

If ν is counting measure on $\mathbf{N}$, then $\mathcal{L}^p(\mathbf{N}, \mathcal{P}(\mathbf{N}), \nu) = L^p(\mathbf{N}, \mathcal{P}(\mathbf{N}), \mu) = \ell^p$. In this case we know that $\|\cdot\|_p$ are complete norms for all $1 \leq p \leq \infty$. More generally, if ν is counting measure on any set X, then we let $\ell^p(X) = L^p(X, \mathcal{P}(X), \nu)$. Then

$$\|f\|_p^p = \int_X |f(x)|^p \, d\nu(x) = \sum_{x \in X} |f(x)|^p$$

where the sum is defined as in HW#30. To see this, just note that a (measurable) simple function is any function vanishing off a finite set F.

Don't Forget Infinity

Definition

If $(X, \mathcal{M}, \mu)$ is a measure space and $f : X \to \mathbf{C}$ is measurable, then

$$\|f\|_\infty = \inf\{\, a \geq 0 : \mu\big(\{\, x : |f(x)| > a \,\}\big) = 0 \,\}$$

with the understanding that $\inf \emptyset := \infty$ in this case. We call $\|f\|_\infty$ the essential supremum of f and sometimes write $\|f\|_\infty = \operatorname{ess\,sup}_{x \in X} |f(x)|$. (The notation overlap with the ordinary "sup norm" is unfortunate, but the notation is classical.)

Remark (The Infimum is Attained)

If $\|f\|_\infty < \infty$, then

$$\{\, x : |f(x)| > \|f\|_\infty \,\} = \bigcup_{n=1}^{\infty} \{\, x : |f(x)| > \|f\|_\infty + \tfrac{1}{n} \,\}. \qquad (*)$$

Thus the LHS of $(*)$ is a null set. Moreover, if $\|f\|_\infty < \epsilon$, then there is a null set N such that $|f(x)| < \epsilon$ if $x \notin N$.

Basic L^∞

Definition

We let $\mathcal{L}^\infty(X, \mathcal{M}, \mu)$ be the collection of measurable functions $f : X \to \mathbf{C}$ such that $\|f\|_\infty < \infty$, and let $L^\infty(X, \mathcal{M}, \mu)$ be the set of almost everywhere equivalence class in $\mathcal{L}^\infty(X, \mathcal{M}, \mu)$.

Proposition

Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. $\|\cdot\|_\infty$ *is a norm on* $L^\infty(\mu)$.
2. $f_n \to f$ *in* $L^\infty(\mu)$ *if and only if there is a* $E \in \mathcal{M}$ *such that* $f_n \to f$ *uniformly on* $X \setminus E$ *and* $\mu(E) = 0$.
3. $L^\infty(\mu)$ *is a Banach space.*
4. *Simple functions are dense in* $L^\infty(\mu)$.

Proof.

This is a homework problem. □

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Conjugate Exponents

Definition

If $1 < p < \infty$, then $q = \frac{p}{p-1}$ is called the conjugate exponent to p. We also declare 1 and ∞ to be conjugate exponents of one another.

Remark

Rather than say "q is the conjugate exponent to p", we will normally just write $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma (HW#1.1+)

If $a, b \in [0, \infty)$ and $0 < \lambda < 1$, then

$$a^{\lambda} b^{1-\lambda} \leq \lambda a + (1-\lambda) b$$

with equality if and only if $a = b$.

▸ return

Hölder's Inequality

Remark (Infinite Norms)

If $f : X \to \mathbf{C}$ *is measurable, and* $\int_X |f(x)|^p \, d\mu(x) = \infty$, *then we will write* $\|f\|_p = \infty$.

Theorem (Hölder's Inequality)

Suppose that $1 \leq p \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $f, g : X \to \mathbf{C}$ *are measurable, then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q. \qquad (\ddagger)$$

In particular, if $f \in \mathcal{L}^p(X)$ *and* $g \in \mathcal{L}^q(X)$, *then* $fg \in \mathcal{L}^1(X)$. *If in addition,* $1 < p < \infty$, *then we have equality in* $(\ddagger)$ *if and only if there are non-negative constants* α *and* β, *not both equal to* 0, *such that* $\alpha|f(x)|^p = \beta|g(x)|^q$ *for* μ*-almost all* x.

Proof

Proof.

This is straightforward if $p = 1$ or $p = \infty$. (When do we get equality in this case?) So assume $1 < p < \infty$. If $\|f\|_p = 0$ or $\|g\|_q = 0$, then $fg \sim 0$ and the result is clear. Hence we can assume $\|f\|_p > 0$ and $\|g\|_q > 0$.

If either $\|f\|_p = \infty$ or $\|g\|_q = \infty$, then the result is clear.

Hence we assume that $0 < \|f\|_p, \|g\|_q < \infty$. Since $\|\cdot\|_p$ and $\|\cdot\|_q$ are homogeneous, the inequality in question amounts to showing

$$\left\| \frac{f}{\|f\|_p} \cdot \frac{g}{\|g\|_q} \right\|_1 \leq 1.$$

Proof Continued.

Therefore we assume that $\|f\|_p = 1 = \|g\|_q$, and it will suffice to prove that $\|fg\|_1 \le 1$ with equality exactly when $|f(x)|^p = |g(x)|^q$ for almost all x.

Let $a = |f(x)|^p$ and $b = |g(x)|^q$ and $\lambda = \frac{1}{p}$. Since $q(1-\lambda) = 1$, our HW Lemma implies

$$|f(x)||g(x)| \le \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q. \qquad (\dagger)$$

Then we integrate to get

$$\|fg\|_1 \le \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q = 1.$$

But we get equality above if and only if we get equality almost everywhere in $(\dagger)$. But this happens at x only if $|f(x)|^p = a = b = |g(x)|^q$. □

Minkowski's Inequality

Theorem (Minkowski's Inequality)

If $1 \le p \le \infty$ and if $f, g \in \mathcal{L}^p(X, \mathcal{M}, \mu)$, then

$$\|f + g\|_p \le \|f\|_p + \|g\|_p.$$

In particular, $\|\cdot\|_p$ is a norm on $L^p(X, \mathcal{M}, \mu)$.

Proof.

The result is easy if $p = 1$ (and we dealt with this case before), and $p = \infty$ is homework. The result is also easy if $f + g \sim 0$. Otherwise

$$\begin{aligned} |f(x) + g(x)|^p &= |f(x) + g(x)|\big(|f(x) + g(x)|\big)^{p-1} \\ &\le |f(x)|\big(|f(x) + g(x)|\big)^{p-1} + |g(x)|\big(|f(x) + g(x)|\big)^{p-1} \end{aligned}$$

Proof

Proof Continued.

Now we apply Hölder with $q = \frac{p}{p-1}$ to get

$$\begin{aligned} \|f+g\|_p^p &\le \int_X |f||f+g|^{p-1}\,d\mu + \int_X |g||f+g|^{p-1}\,d\mu \\ &\le \|f\|_p \||f+g|^{p-1}\|_q + \|g\|_p \||f+g|^{p-1}\|_q. \end{aligned}$$

But

$$\||f+g|^{p-1}\|_q = \Bigl(\int_X |f+g|^p\Bigr)^{\frac{1}{q}} = \|f+g\|_p^{\frac{p}{q}}.$$

Thus

$$\|f+g\|_p^{p-\frac{p}{q}} \le \|f\|_p + \|g\|_p$$

and $p - \frac{p}{q} = 1$. □

Theorem

If $(X, \mathcal{M}, \mu)$ is a measure space, then $L^p(X, \mathcal{M}, \mu)$ is a Banach space for $1 \le p \le \infty$.

Proof.

We did the case $p = 1$ earlier in the course and $p = \infty$ is homework. So suppose $1 < p < \infty$. As in the case $p = 1$, it will suffice to see that an absolutely convergent series in convergent. So suppose $\{ f_k \} \subset \mathcal{L}^p(X)$ and suppose

$$\sum_{k=1}^{\infty} \|f_k\|_p = B < \infty.$$

Let $g_n(x) = \sum_{k=1}^{n} |f_k(x)|$ and $g(x) = \sum_{k=1}^{\infty} |f_k(x)|$.

Proof Continued.

By Minkowski,

$$\|g_n\|_p \leq \sum_{k=1}^{n} \|f_k\|_p \leq B.$$

By the MCT,

$$\int_X g(x)^p \, d\mu(x) = \lim_n \int_X g_n(x)^p = \lim_n \|g_n\|_p^p \leq B^p < \infty.$$

Therefore $g \in \mathcal{L}^p(X)$ and we must have $g(x) < \infty$ for almost all x. Since $\mathbf{C}$ is complete and $\sum_{k=1}^{\infty} f_k(x)$ is absolutely convergent almost everywhere,

$$f(x) = \begin{cases} \sum_{k=1}^{\infty} f_k(x) & \text{if the series converges, and} \\ 0 & \text{otherwise} \end{cases}$$

is a measurable function.

Proof

Proof Continued.

Since $|f(x)| \leq \sum_{k=1}^{\infty} |f_k(x)| = g(x)$, $f \in \mathcal{L}^p(X)$. Furthermore,

$$\Big| f(x) - \sum_{k=1}^{n} f_k(x) \Big|^p \leq 2^p g(x)^p.$$

Since $2^p g^p \in \mathcal{L}^1(X)$, the LDCT implies

$$\lim_{n\to\infty} \int_X \Big| f(x) - \sum_{k=1}^{n} f_k(x) \Big|^p \, d\mu(x) = 0.$$

Thus

$$\| f - \sum_{k=1}^{n} f_k \|_p \to 0$$

as required. □

That's Enough for Today

- That is enough for now.