Math 73/103: Fall 2020
Lecture 25

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Getting Started

- We should be recording!
- Questions?
- We will have one more homework assignment due on Tuesday, November 17th.
- There will be a final exam, really a "final homework set" due by the end of the day on Monday, November 30th.
- When should I "release it"?

Loose Ends

Proposition

If $(X, \mathcal{M}, \mu)$ is a measure space, then integrable simple functions are dense in $L^p(X, \mathcal{M}, \mu)$ for $1 \leq p < \infty$.

Proposition

Integrable step functions are dense in $L^p(\mathbf{R}, \mathcal{L}, m)$ for $1 \leq p < \infty$.

Proposition

$C_c(\mathbf{R})$ is dense in $L^p(\mathbf{R}, \mathcal{L}, m)$ for $1 \leq p < \infty$.

Proof.

I leave these to you as well figuring out what happens in the case $p = \infty$. □

The Heck with Measure Completions

Remark

Suppose that $(X, \mathcal{M}, \mu)$ is a measure space, and let $(X, \mathcal{M}_0, \mu_0)$ be its completion. Suppose $1 \leq p \leq \infty$. Recall that if $f : X \to \mathbf{C}$ is μ-measurable, then it is μ_0-measurable. Furthermore, the map sending the equivalence class $[f] \in L^p(X, \mathcal{M}, \mu)$ to the equivalence class $[f]_0$ of f in $L^p(X, \mathcal{M}_0, \mu_0)$ is surjective by HW#39 as well as norm-preserving—we would say isometric. Therefore $[f] \mapsto [f]_0$ induces an isometric isomorphism of $L^p(X, \mathcal{M}, \mu)$ onto $L^p(X, \mathcal{M}_0, \mu_0)$. For example, there is no difference as Banach spaces between $L^p(\mathbf{R}, \mathcal{B}(\mathbf{R}), m)$ and $L^p(\mathbf{R}, \mathcal{L}, m)$. Alternatively, given $[f] \in L^p(\mathbf{R})$, we can always assume f is Borel if we wish.

Looking Ahead to Functional Analysis

Definition

If V is a normed (complex) vector space, then a linear map $\varphi : V \to \mathbf{C}$ is called a linear functional. We say that φ is bounded if

$$\|\varphi\| := \sup_{\|v\|\leq 1} |\varphi(v)| < \infty.$$

The set V^* of bounded linear functionals is called the dual of V.

Example

If V is finite dimensional with basis $\{ v_1, \dots, v_n \}$, then you will see in Math 113 that every linear functional is bounded and that V^* is finite dimensional with dual basis $\{ v_1^*, \dots, v_n^* \}$ where $v_k^*(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_k$. The situation is much more complex in infinite dimensions.

Continuity Implies Bounded

Proposition

Let φ be a linear functional on V. If φ is bounded, then φ is continuous and

$$|\varphi(v)| \leq \|\varphi\|\|v\|. \tag{$*$}$$

Conversely, if φ is continuous, then φ is bounded. In fact, it suffices for φ to be continuous at $0 \in V$.

Proof.

Suppose $\varphi \in V^*$. Then for all $v \in V \setminus \{0\}$, we have

$$|\varphi(v)| = \Big|\varphi\Big(\frac{v}{\|v\|}\Big)\Big|\|v\| \leq \|\varphi\|\|v\|.$$

This proves $(*)$. But then

$$|\varphi(v) - \varphi(w)| = |\varphi(v - w)| \leq \|\varphi\|\|v - w\|,$$

and φ is continuous—uniformly continuous in fact.

Proof

Proof Continued.

Now suppose that φ is continuous at 0. Then $\varphi^{-1}(B_1^{\mathbf{C}}(0))$ is a neighborhood of 0 in V. Thus there is an $\epsilon > 0$ such that $B_{2\epsilon}^V(0) \subset \varphi^{-1}(B_1^{\mathbf{C}}(0))$. Now if $\|v\| \le 1$, $\epsilon v \in B_{2\epsilon}^V(0)$ and we have

$$|\varphi(v)| = \frac{1}{\epsilon}|\varphi(\epsilon v)| \le \frac{1}{\epsilon}.$$

Since $\|v\| \le 1$ was arbitrary, $\|\varphi\| \le \frac{1}{\epsilon} < \infty$. □

Example

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Fix $g \in \mathcal{L}^q(X)$ and define $\varphi_g : \mathcal{L}^p(X) \to \mathbf{C}$ by

$$\varphi_g(f) = \int_X f(x)g(x)\,d\mu(x).$$

This makes sense because

$$|\varphi_g(f)| \leq \int_X |f(x)g(x)|\,d\mu(x) = \|fg\|_1 \leq \|f\|_p \|g\|_q < \infty.$$

More to the point, we can view φ_g as a linear functional on $L^p(X)$—technically sending $[f] \mapsto \varphi_g(f)$—and this functional is bounded with $\|\varphi_g\| \leq \|g\|_q$. As usual, we ignore equivalence classes of functions whenever possible.

Isometric

Definition

If $(X, \mathcal{M}, \mu)$ is a measure space, then we say that μ is semifinte if whenever $E \in \mathcal{M}$ and $\mu(E) = \infty$, then there is a $K \subset E$ such that $0 < \mu(K) < \infty$.

Remark

σ-finite measures are semifinite, but the converse can fail.

Proposition

Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and that $1 \leq q < \infty$. Then for all $g \in \mathcal{L}^q(X)$,

$$\|\varphi_g\| = \|g\|_q. \tag{\dagger}$$

If μ is semifinite, then $(\dagger)$ holds for $q = \infty$ as well.

Proof

Proof.

We already know that $\|\varphi_g\| \leq \|g\|_q$. We certainly have equality if $g \sim 0$, so we can assume $\|g\|_q > 0$. Let $1 < q < \infty$ and define

$$\operatorname{sgn}(z) = \begin{cases} \frac{z}{|z|} & \text{if } z \in \mathbf{C} \setminus \{0\}, \text{ and} \\ 0 & \text{if } z = 0. \end{cases}$$

Note that $g(x) = |g(x)| \operatorname{sgn}(g(x))$ for all $x \in X$. Then let

$$f(x) = \|g\|_q^{1-q} |g(x)|^{q-1} \overline{\operatorname{sgn}(g(x))}.$$

Then ▸ return

$$\begin{aligned} \|f\|_p^p &= \|g\|_q^{p-pq} \int_X |g(x)|^{pq-p} \, d\mu(x) \\ &= \|g\|_q^{-q} \int_X |g(x)|^q \, d\mu(x) = 1. \end{aligned}$$

Proof

Proof Continued.

Therefore

$$\|\varphi_g\| \geq |\varphi_g(f)| = \|g\|_q^{1-q} \int_X |g(x)|^q \, d\mu(x) = \|g\|_q.$$

This takes care of $1 < q < \infty$. If $q = 1$, then we can let $f(x) = \overline{\operatorname{sgn}(g(x))}$. Then $\|f\|_\infty = 1$ and

$$\varphi_g(f) = \int_X |g(x)| \, d\mu(x) = \|g\|_1.$$

This takes care of $q = 1$.

Proof

Proof Continued.

Now suppose $q = \infty$. Let $\epsilon > 0$ and set $A = \{ x : |g(x)| > \|g\|_\infty - \epsilon \}$. By definition of the essential supremum, $\mu(A) > 0$. If μ is semifinite, then there is a $B \subset A$ such that $0 < \mu(B) < \infty$. Let

$$f(x) = \mu(B)^{-1}\overline{\operatorname{sgn}(g(x))}\mathbb{1}_B(x).$$

Now $\|f\|_1 = 1$ and

$$\begin{aligned} \|\varphi_g\| \geq |\varphi_g(f)| &= \Bigl| \int_X f(x)g(x)\,d\mu(x) \Bigr| \\ &= \frac{1}{\mu(B)} \int_B |g(x)|\,d\mu(x) \geq \|g\|_\infty - \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\|\varphi_g\| \geq \|g\|_\infty$. This completes the proof. □

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Technicalities

Corollary

If $(X, \mathcal{M}, \mu)$ is a measure space, then the map $g \mapsto \varphi_g$ is an isometric linear injection of $L^q(X)$ into $L^p(X)^*$ for $1 \leq q < \infty$. If μ is semifinite, then this holds for $q = \infty$ as well.

Notation

For a while, $(X, \mathcal{M}, \mu)$ will be an arbitrary measure space. We let Σ be the complex vector space of integrable simple functions on X. Note that if $f \in \Sigma$, then

$$f = \sum_{k=1}^{n} a_k \mathbb{1}_{E_k}$$

with $\mu(E_k) < \infty$ for all $1 \leq k \leq n$. In particular, $\Sigma \subset \mathcal{L}^p(X)$ for all $1 \leq p \leq \infty$.

Simple Functions

Remark

Suppose that $f : X \to \mathbf{C}$ *is measurable. Then* $f = u_1 - u_2 + i(u_3 - u_4)$ *where each* $u_k : X \to [0, \infty)$ *is measurable with* $u_1 u_2 = 0$ *and* $u_3 u_4 = 0$. *Then there are MNNSFs* $s_{n,k} \nearrow u_k$. *Hence* $f_n = s_{n,1} - s_{n,2} + i(s_{n,3} - s_{n,4})$ *is a simple function and* (f_n) *converges pointwise to* f. *Note that for each* n, $s_{n,1} \cdot s_{n,2} = 0$ *and* $s_{n,3} \cdot s_{n,4} = 0$. *It follows that* $|\operatorname{Re}(f_n)| \leq |\operatorname{Re}(f)|$ *and* $|\operatorname{Im}(f_n)| \leq |\operatorname{Im}(f)|$. *Hence* $|f_n| \leq |f|$.

Technical Proposition

Proposition

Suppose that $g : X \to \mathbf{C}$ *is measurable and that* $fg \in \mathcal{L}^1(X)$ *for all* $f \in \Sigma$. *If* $\frac{1}{p} + \frac{1}{q} = 1$, *then let*

$$M_q(g) := \sup\{\, |\varphi_g(f)| : f \in \Sigma \text{ and } \|f\|_p = 1 \,\}.$$

If $M_q(g) < \infty$ *and if either*

1. $S_g := \{\, x : |g(x)| > 0 \,\}$ *is* σ*-finite, or*
2. μ *is semifinite,*

then $g \in \mathcal{L}^q(X)$ *and* $M_q(g) = \|g\|_q$.

Lemma

Let g and $M_q(g)$ be as in the Proposition. Suppose that f is a bounded measurable function that vanishes off a set E of finite measure such that $\|f\|_p = 1$. Then

$$\Big|\int_X f(x)g(x)\,d\mu(x)\Big| \le M_q(g).$$

return

Proof.

By our earlier remark, there are measurable simple functions f_n such that $f_n \to f$ pointwise with $|f_n| \le |f|$. Then each f_n vanishes off E and $|f_n| \le \|f\|_\infty \mathbb{1}_E$. Since $\|f\|_\infty \mathbb{1}_E \cdot g \in \mathcal{L}^1(X)$ by assumption on g, the LDCT implies,

$$\Big|\int_X fg\,d\mu\Big| = \lim_n \Big|\int_X f_n g\,d\mu\Big| \le M_q(g). \qquad \square$$

Proof of the Proposition.

Assume $q < \infty$. We will show in HW#51 that if $M_q(g) < \infty$ and μ is semifinite, then S_g is σ-finite. Hence for $q < \infty$, we will proceed under the assumption that S_g is σ-finite. Suppose $S_g = \bigcup_n E_n$ with $E_n \subset E_{n+1}$ and $\mu(E_n) < \infty$ for all n. Let (φ_n) be a sequence of measurable simple functions such that $\varphi_n \to g$ pointwise and $|\varphi_n| \leq |g|$. Let $g_n = \varphi_n \cdot \mathbb{1}_{E_n}$. Then $g_n \to g$ pointwise, $|g_n| \leq |g|$, and g_n vanishes off E_n. Let

$$f_n(x) = \|g_n\|_q^{1-q} \cdot |g_n(x)|^{q-1} \cdot \overline{\operatorname{sgn}(g(x))}.$$

Then f_n is a bounded function vanishing off E_n and we can compute, just as we did earlier, that $\|f_n\|_p = 1$. Furthermore,

$$\int_X |f_n(x)g_n(x)|\,d\mu(x) = \|g_n\|_q^{1-q} \int_X |g_n(x)|^q\,d\mu(x) = \|g_n\|_q.$$

Proof Continued.

By Fatou's Lemma

$$\|g\|_q^q = \int_X |g(x)|^q \, d\mu(x) \leq \liminf_n \int_X |g_n(x)|^q \, d\mu(x) = \liminf_n \|g_n\|_q^q.$$

Hence

$$\begin{aligned} \|g\|_q &\leq \liminf_n \|g_n\|_q = \liminf_n \int_X |f_n g_n| \, d\mu \\ &\leq \liminf \int_X |f_n g| \, d\mu = \liminf_n \int_X f_n g \, d\mu \\ &\leq \liminf_n \Big| \int_X f_n g \, d\mu \Big| \leq M_q(g) \end{aligned}$$

where we used our lemma for the last inequality. Our assumption that $M_q(g) < \infty$ shows that $g \in \mathcal{L}^q(X)$. Then Hölder implies that $M_q(g) \leq \|g\|_q$. Therefore we have proved the result when $q < \infty$.

Proof

Proof Continued.

Now assume $q = \infty$ and let $\epsilon > 0$. Let

$$A = \{\, x : |g(x)| \geq M_\infty(g) + \epsilon \,\}.$$

Suppose that $\mu(A) > 0$. Note that $A \subset S_g$. Hence if either μ is semifinite or if S_g is σ-finite, there is a $B \subset A$ such that $0 < \mu(B) < \infty$. Let $f = \mu(B)^{-1}\overline{\operatorname{sgn}(g)}\mathbb{1}_B$. Then $\|f\|_1 = 1$, and

$$\varphi_g(f) = \int_X fg \, d\mu = \frac{1}{\mu(B)} \int_B |g| \, d\mu \geq M_\infty(g) + \epsilon.$$

But f is a bounded function vanishing off a set of finite measure so our lemma implies $|\varphi_g(f)| < M_\infty(g)$ which leads to a contradiction. Hence $\mu(A) = 0$ and $\|g\|_\infty \leq M_\infty(g)$. Just as above, this implies $g \in \mathcal{L}^\infty(X)$ and then Hölder implies $M_\infty(g) = \|g\|_\infty$. □

Preview of Coming Attactions

Theorem

Suppose that $(X, \mathcal{M}, \mu)$ is any measure space and $\frac{1}{p} + \frac{1}{q} = 1$. If $1 < p < \infty$, then $g \mapsto \varphi_g$ is an isometric isomorphism of $L^q(X)$ onto $L^p(X)^*$. If μ is σ-finite, then $g \mapsto \varphi_g$ is an isometric isomorphism of $L^\infty(X)$ onto $L^1(X)^*$.

Remark

We will prove this Wednesday employing our technical proposition. I find it remarkable that there are no assumptions on $(X, \mathcal{M}, \mu)$ when $1 < p < \infty$. The restriction to σ-finiteness is necessary when $p = 1$. There isn't much useful to say about $L^\infty(X)^*$ except for trivial special cases.

That's Enough for Today

- That is enough for now.