Math 73/103: Fall 2020
Lecture 26

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Getting Started

- We should be recording!
- Questions?
- Our last homework assignment—problems #46–#54—are due Tuesday via `gradescope`.
- The final exam, really a "final homework set" is due by the end of the day on Monday, November 30th and will be turned in via `gradescope`.
- I will send the exam out via email. When should I "release it"?

Last Time

Proposition (Technical Result)

Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and that Σ is the subspace of integrable measurable simple functions. Suppose that $g : X \to \mathbf{C}$ is measurable and that $fg \in \mathcal{L}^1(X)$ for all $f \in \Sigma$. If $\frac{1}{p} + \frac{1}{q} = 1$, then let

$$M_q(g) := \sup\{\, |\varphi_g(f)| : f \in \Sigma \text{ and } \|f\|_p = 1 \,\}.$$

If $M_q(g) < \infty$ and if either

1. $S_g := \{\, x : |g(x)| > 0 \,\}$ *is σ-finite, or*
2. μ *is semifinite,*

then $g \in \mathcal{L}^q(X)$ and $M_q(g) = \|g\|_q$.

Mea Culpa

Lemma

Let g and $M_q(g)$ be as in the Proposition. Suppose that f is a bounded measurable function that vanishes off a set E of finite measure such that $\|f\|_p = 1$. Then

$$\Big| \int_X f(x)g(x)\,d\mu(x) \Big| \le M_q(g).$$

Proof.

By our earlier remark, there are measurable simple functions f_n such that $f_n \to f$ pointwise with $|f_n| \le |f|$. Then each f_n vanishes off E and $|f_n| \le \|f\|_\infty \mathbb{1}_E$. Since $\|f\|_\infty \mathbb{1}_E \cdot g \in \mathcal{L}^1(X)$ by assumption on g, the LDCT implies,

$$\Big| \int_X fg\,d\mu \Big| = \lim_n \Big| \int_X f_n g\,d\mu \Big| \le M_q(g). \quad \square$$

Radon Nikodym Revisited

Theorem (Radon Nikodym for Complex Measures)

Suppose that $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and that ν is a complex measure on $(X, \mathcal{M})$ such that $\nu(E) = 0$ whenever $\mu(E) = 0$. Then there is a $f \in \mathcal{L}^1(\mu)$ such that

$$\nu(E) = \int_E f(x)\, d\mu(x) \quad \text{for all } E \in \mathcal{M},$$

and f is determined up to a μ null set.

Remark

We could write $\nu \ll \mu$ and $f = \frac{d\nu}{d\mu}$, but we will reserve these notations for (positive) measures in this course.

Proof

Proof.

To start with, assume ν is a real-valued measure with Jordan decomposition $\nu = \nu^+ - \nu^-$. Then $\nu^\pm$ are mutually singular finite (positive) measures. Therefore we have a partition $X = P \cup N$ so that $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = \nu(E \cap N)$. Suppose $\mu(E) = 0$. Then $\mu(E \cap P) = 0$, and hence by assumption, $\nu(E \cap P) = 0$. But then $\nu^+(E) = \nu^+(E \cap P) = \nu(E \cap P) = 0$. Therefore $\nu^+ \ll \mu$. By the Radon-Nikodym Theorem, there is a measurable function $f^+ : X \to [0, \infty)$ such that

$$\nu^+(E) = \int_E f^+(x)\,d\mu(x) \quad \text{for all } E \in \mathcal{M}. \tag{1}$$

Since $\nu^+(X) < \infty$, $f^+ \in \mathcal{L}^1(\mu)$.

Proof Continued.

Similarly, $\nu^- \ll \mu$ and there is a $f^- \in \mathcal{L}^1(\mu)$ such that (1) holds for ν^-. Hence

$$\nu(E) = \int_E f(x)\,d\mu(x) \quad \text{for all } E \in \mathcal{M}$$

where $f = f^+ - f^- \in \mathcal{L}^1(\mu)$.

We get the full result by writing $\nu = \operatorname{Re}(\nu) + i \operatorname{Im}(\nu)$ and observing that the above applies to the real-valued measures $\operatorname{Re}(\nu)$ and $\operatorname{Im}(\nu)$.

Uniqueness is determined exactly as before. $\square$

Last Time

We have the following key result from the previous lecture:

Corollary (Isometric)

Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and that $\frac{1}{p} + \frac{1}{q} = 1$. If $g \in \mathcal{L}^q(X)$, then we define $\varphi_g : \mathcal{L}^p(X) \to \mathbf{C}$ by

$$\varphi_g(f) = \int_X f(x)g(x)\,d\mu(x).$$

If $1 \leq q < \infty$, then $g \mapsto \varphi_g$ is an isometric linear injection of $L^q(X)$ into $L^p(X)^*$. If μ is semifinite, then this holds for $q = \infty$ and $p = 1$ as well.

Uniqueness

Corollary (Uniqueness)

If g and g' are in $\mathcal{L}^q(X, \mathcal{M}, \mu)$ and $\varphi_g = \varphi_{g'}$, then $g \sim g'$ if $1 \leq q < \infty$ and also for $q = \infty$ if μ is semifinite.

Proof.

If $\varphi_g = \varphi_{g'}$, then $\varphi_{g-g'} = 0$ and $\|g - g'\|_q = 0$. □

The Big Theorem

Theorem

Suppose that $(X, \mathcal{M}, \mu)$ is any measure space and $\frac{1}{p} + \frac{1}{q} = 1$. If $1 < p < \infty$, then $g \mapsto \varphi_g$ is an isometric isomorphism of $L^q(X)$ onto $L^p(X)^*$. If μ is σ-finite, then $g \mapsto \varphi_g$ is an isometric isomorphism of $L^\infty(X)$ onto $L^1(X)^*$.

Proof.

In view of our corollary from last lecture, it will suffice to show that if $\varphi \in L^p(X)^*$, then there is a $g \in \mathcal{L}^q(X)$ such that $\varphi = \varphi_g$.

To start with, we assume $1 \leq p < \infty$ and $\mu(X) < \infty$.

Proof

Proof Continued.

Since $\mu(X) < \infty$, $\mathbb{1}_E \in \mathcal{L}^p(X)$ for all $E \in \mathcal{M}$. Hence we can define

$$\nu : \mathcal{M} \to \mathbf{C} \quad \text{by} \quad \nu(E) = \varphi(\mathbb{1}_E).$$

Suppose that $E \in \mathcal{M}$ is the disjoint union $\bigcup E_n$. Then

$$\Bigl\| \mathbb{1}_E - \sum_{k=1}^{n} \mathbb{1}_{E_k} \Bigr\|_p = \Bigl\| \sum_{k=n+1}^{\infty} \mathbb{1}_{E_k} \Bigr\|_p = \mu\Bigl(\bigcup_{k=n+1}^{\infty} E_k \Bigr)^{\frac{1}{p}}$$

which tends to 0 since μ is finite.

That is

$$\mathbb{1}_E = \sum_{n=1}^{\infty} \mathbb{1}_{E_n} \quad \text{in } L^p(X).$$

Proof Continued.

Then, since φ is continuous,

$$\nu(E) = \varphi(\mathbb{1}_E) = \sum_{n=1}^{\infty} \varphi(\mathbb{1}_{E_n}) = \sum_{n=1}^{\infty} \nu(E_n).$$

Therefore ν is a complex measure on $(X, \mathcal{M})$. Furthermore, if $\mu(E) = 0$, then $\mathbb{1}_E = 0$ in $L^p(X)$ and $\nu(E) = \varphi(\mathbb{1}_E) = 0$. (That is, "$\nu \ll \mu$".) Hence, by our Radon-Nikodym Theorem for complex measures, there is a $h \in \mathcal{L}^1(\mu)$ such that

$$\varphi(\mathbb{1}_E) = \nu(E) = \int_E h(x)\,d\mu(x) = \int_X \mathbb{1}_E(x)h(x)\,d\mu(x)$$

for all $E \in \mathcal{M}$.

Proof

Proof Continued.

Consequently,

$$\varphi(f) = \int_X f(x)h(x)\,d\mu(x)$$

for all (necessarily integrable) simple functions f. Since φ is bounded,

$$\Bigl|\int_X f(x)h(x)\,d\mu(x)\Bigr| \le \|f\|_p\|\varphi\| \le \|\varphi\| \quad \text{for all } f \in \Sigma.$$

Thus $M_q(h) < \infty$ and our technical proposition implies that $h \in \mathcal{L}^q(X)$.

Since simple functions are dense in $L^p(X)$ and since φ and φ_h agree on simple functions, we have $\varphi = \varphi_h$. Since $\|\varphi\| = \|h\|_q$, it follows that h is determined μ-almost everywhere.

This completes the proof in the case μ is finite.

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Proof in the σ-Finite Case

Proof Continued.

Now we suppose that μ is σ-finite and $1 \leq p < \infty$. Say $X = \bigcup E_n$ with $\mu(E_n) < \infty$ and $E_n \subset E_{n+1}$. Let $\mu_n(E) = \mu(E \cap E_n)$. As we observed in our proof of the σ-finite case of the Radon-Nikodym Theorem in Lecture 22, for $f \geq 0$ we have

$$\int_X f(x)\, d\mu_n(x) = \int_X \mathbb{1}_{E_n}(x) f(x)\, d\mu(x).$$

Then if we let $\varphi_n : L^p(X, \mu_n) \to \mathbf{C}$ be given by $\varphi_n(f) = \varphi(\mathbb{1}_{E_n} \cdot f)$, then

$$|\varphi_n(f)| \leq \|\varphi\| \|\mathbb{1}_{E_n} \cdot f\|_{L^p(\mu)} = \|\varphi\| \|f\|_{L^p(\mu_n)}$$

Proof

Proof Continued.

Thus $\varphi_n \in L^p(\mu_n)^*$ with $\|\varphi_n\| \le \|\varphi\|$, and there is a $h_n \in \mathcal{L}^q(X, \mu_n)$ such that

$$\varphi(\mathbb{1}_{E_n} \cdot f) = \varphi_n(f) = \int_X f(x) h_n(x)\, d\mu_n(x)$$
$$= \int_X \mathbb{1}_{E_n}(x) f(x) h_n(x)\, d\mu(x).$$

Hence we can assume $h_n(x) = 0$ if $x \notin E_n$. Furthermore, if $n < m$, then $h_m(x) = h_n(x)$ for μ-almost all $x \in E_n$. Let $g_n = \max\{ h_1, \dots, h_n \}$ and note that $g_n = h_n$ μ-almost everywhere.

Let $g = \sup_n g_n = \lim g_n$.

Proof Continued.

Thus if $1 < p < \infty$, then $1 < q < \infty$ and

$$\int_X |g|^q \, d\mu \stackrel{\text{MCT}}{=} \lim_n \int_X |g_n|^q \, d\mu \leq \|\varphi\|^q.$$

Thus $g \in \mathcal{L}^q(X)$. If $p = 1$ and $q = \infty$, then $\|g_n\|_\infty \leq \|\varphi\|$ for all n implies $\|g\|_\infty \leq \|\varphi\|$, and $g \in \mathcal{L}^\infty(X)$.

Now if $f \in \mathcal{L}^p(X)$ for $1 \leq p < \infty$, then $\mathbb{1}_{E_n} \cdot f \to f$ pointwise and $|\mathbb{1}_{E_n} \cdot f - f|^p \leq 2^p |f|^p \in \mathcal{L}^1(X)$, so

$$\mathbb{1}_{E_n} \cdot f \to f \quad \text{in } L^p(X) \text{ by the LDCT.}$$

Proof

Proof Continued.

Therefore if $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q(X)$ as above, then $fg \in \mathcal{L}^1(X)$ and

$$\begin{aligned}\varphi(f) &= \lim_n \varphi(\mathbb{1}_{E_n} \cdot f) = \lim_n \int_X \mathbb{1}_{E_n}(x) f(x) h_n(x)\, d\mu(x) \\ &= \lim_n \int_X \mathbb{1}_{E_n}(x) f(x) g_n(x)\, d\mu(x) \\ &\overset{\text{LDCT}}{=} \int_X f(x) g(x)\, d\mu(x).\end{aligned}$$

That is, $\varphi = \varphi_g$ and we have established the result in the σ-finite case for $1 \leq p < \infty$.

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Proof in the General Case.

We now make no assumptions on μ, but we restrict to $1 < p < \infty$. Let $E \subset X$ be σ-finite and let $\mu_E(A) = \mu(A \cap E)$. Then μ_E is a σ-finite measure on $(X, \mathcal{M})$ such that for all $f \geq 0$

$$\int_X f(x)\, d\mu_E(x) = \int_X \mathbb{1}_E(x) f(x)\, d\mu(x).$$

Again we define $\varphi_E(f) = \varphi(\mathbb{1}_E f)$. Then

$$|\varphi_E(f)| = |\varphi(\mathbb{1}_E f)| \leq \|\varphi\| \|\mathbb{1}_E f\|_{L^p(\mu)} = \|\varphi\| \|f\|_{L^p(\mu_E)}.$$

Therefore $\varphi_E \in L^p(X, \mu_E)^*$ and there is a $g_E \in \mathcal{L}^q(X, \mu_E)$ such that

$$\varphi_E(f) = \int_X f(x) g_E(x)\, d\mu_E(x) = \int_X \mathbb{1}_E(x) f(x) g_E(x)\, d\mu(x).$$

Proof

Proof Continued.

We can assume $g_E(x) = 0$ if $x \notin E$. Thus we can write $\|g_E\|_q$ unambiguously. Note that $\|\varphi_E\| = \|g_E\|_q \leq \|\varphi\|$. In particular, g_E is determined μ-almost everywhere on E. Thus if $F \supset E$ is also σ-finite, then $g_F = g_E$ μ-almost everywhere on E and $\|g_F\|_q \geq \|g_E\|_q$.

Let

$$M = \sup\{\, \|g_E\|_q : E \text{ is } \sigma\text{-finite} \,\}.$$

We have $M \leq \|\varphi\| < \infty$. Let $\{\, E_n \,\}$ be σ-finite subsets such that $\|g_{E_n}\|_q \to M$. Then $F = \bigcup_n E_n$ is σ-finite and $M \geq \|g_F\|_q \geq \|g_{E_n}\|_q$ for all n. Hence $\|g_F\|_q = M$.

Proof

Proof Continued.

If $A \supset F$ is σ-finite, then

$$g_A = g_F + g_{A \setminus F}$$

almost everywhere. Since $q < \infty$,

$$\int_X |g_F|^q \, d\mu + \int_X |g_{A \setminus F}|^q \, d\mu = \int_X |g_A|^q \, d\mu \leq M^q = \int_X |g_F|^q \, d\mu$$

Therefore $\|g_{A \setminus F}\|_q = 0$ and $g_{A \setminus F} = 0$ almost everywhere so that $g_A = g_F$ almost everywhere.

Proof

Proof Continued.

Thus if $f \in \mathcal{L}^p(X)$, then $A = F \cup \{ x : |f(x)| > 0 \}$ is σ-finite and

$$\begin{aligned}\varphi(f) = \varphi(\mathbb{1}_A f) &= \int_X f(x) g_A(x)\, d\mu(x) \\ &= \int_X f(x) g_F(x)\, d\mu(x).\end{aligned}$$

Therefore, $\varphi = \varphi_{g_F}$ and we're done. □

That's Enough for Today

- That is enough for now.