Math 73/103: Fall 2020
Lecture 27

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Friday, November 13, 2020

Getting Started

- We should be recording!
- Questions?

Review of Duality for $\mathcal{L}^p$

Remark

Let $(X, \mathcal{M}, \mu)$ be a measure space. We showed that if $1 < p < \infty$, then $g \mapsto \varphi_g$ is a isometric Banach space isomorphism of $L^q(X)$ onto $L^p(X)^*$ where

$$\varphi_g(f) := \int_X f(x)g(x)\,d\mu(x) \quad \text{for all } f \in \mathcal{L}^p(X).$$

Remark

If $p = \infty$, then $g \mapsto \varphi_g$ is an isometric injection of $L^1(X)$ into $L^\infty(X)^*$, but this map never surjective except for trivial special cases.

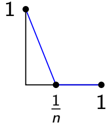
An Example

Example (Where $g \mapsto \varphi_g$ is not onto $L^\infty(X)$)

- Let $X = [0, 1]$ and let $\mu = m$ be Lebesgue measure.
- Then we can view $C([0, 1])$ as a subspace of $L^\infty([0, 1])$: the map $f \in C([0, 1]) \mapsto [f] \in L^\infty([0, 1])$ is an isometric linear map.
- The map $f \mapsto f(0)$ is a bounded linear functional on $C([0, 1])$. In fact, this functional has norm 1.
- In Math 113, we will learn that every bounded linear functional on a subspace M of a normed vector space V has a norm preserving extension to the whole vector space V. This is called the Hahn-Banach Theorem.
- This means there must be a $\varphi \in L^\infty([0, 1])^*$ such that $\varphi(f) = f(0)$ for all $f \in C([0, 1])$.

Example Continued

- Suppose that there were some $g \in \mathcal{L}^1([0,1])$ such that
$$\varphi(f) = \varphi_g(f) = \int_0^1 f(x)g(x)\,dx \quad \text{for all } f \in L^\infty([0,1]).$$
- Let $f_n \in C([0,1])$ be the function with graph



- But $f_n g \to 0$ almost everywhere and $|f_n g| \le g \in \mathcal{L}^1([0,1])$. Hence
$$\varphi(f_n) = \int_0^1 f_n g\,dx \to 0$$
by the LDCT. But $\varphi(f_n) = 1$ for all n. Hence no such g can exist.

$p = 1$

We want to consider the map $g \mapsto \varphi_g$ from $L^\infty(X)$ to $L^1(X)^*$.

Remark (Injectivity)

If μ is semifinite, we proved this map was isometric, and hence injective. If μ is not semifinite, then there is a $F \in \mathcal{M}$ such that $\mu(F) = \infty$ and such that F has no subsets of strictly positive finite measure. Note that $\|\mathbb{1}_F\|_\infty = 1$. Consider $\varphi = \varphi_{\mathbb{1}_F}$. If $E \in \mathcal{M}$ has finite measure, then $\varphi(\mathbb{1}_E) = \int_X \mathbb{1}_E \cdot \mathbb{1}_F \, d\mu = \mu(E \cap F) = 0$. Therefore $\varphi(f) = 0$ for any integrable simple function. Since integrable simple functions are dense in $L^1(X)$, $\varphi = 0$. Thus $g \mapsto \varphi_g$ is injective if and only if μ is semifinite.

Surjectivity

Remark (Surjectivity)

We proved that $g \mapsto \varphi_g$ *is surjective from* $L^\infty(X)$ *to* $L^1(X)^*$ *if* μ *is* σ*-finite. This can fail if* μ *is not* σ*-finite. Let* ν *be counting measure on* $(\mathbf{R}, \mathcal{P}(\mathbf{R}))$*. Let* $\mathcal{M} = \{\, E \subset \mathbf{R} : \text{either } E \text{ or } E^C \text{ is countable} \,\}$*. Let* μ *be the restriction of* ν *to* $\mathcal{M}$*. Then both* μ *and* ν *are semifinite measures that are not* σ*-finite. If* $f \in \mathcal{L}^1(\nu) = L^1(\nu)$*, then* f *vanishes off a countable set. Thus* f *is* $\mathcal{M}$*-measurable since* $f^{-1}(V)$ *is either countable or co-countable for any open set* $V \subset \mathbf{C}$ *(depending on whether or not* $0 \in V$*). Thus* $L^1(\nu) = L^1(\mu)$*. Clearly,* $L^\infty(\nu)$ *is the set* $\ell^\infty(\mathbf{R})$ *of all bounded functions on* $\mathbf{R}$*. Since* $\mathcal{M}$*-measurable functions are constant off a countable set,* $L^\infty(\mu)$ *is the proper subset of* $L^\infty(\nu)$ *of bounded functions which are constant off a countable set. Since* ν *is semifinite,* $g \mapsto \varphi_g$ *is an injection of* $L^\infty(\nu)$ *into* $L^1(\nu)^* = L^1(\mu)^*$*. Therefore* $g \mapsto \varphi_g$ *is not a surjection of* $L^\infty(\mu) \subsetneq L^\infty(\nu)$ *onto* $L^1(\mu)^*$*.*

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Lemma

Let $(\mathbf{R}, \mathcal{L}, m)$ be Lebesgue measure. If $E \in \mathcal{L}$ and $y \in \mathbf{R} \setminus \{0\}$, then let

$$E + s = \{\, r + s : r \in E \,\} \quad \text{and} \quad sE = \{\, sr : r \in E \,\}.$$

Then for all $s \in \mathbf{R}$, $E + s$ and sE are in $\mathcal{L}$. Furthermore, $m(E + s) = m(E)$ and $m(sE) = |s| m(E)$.

Proof.

We proved the translation Invariance back in Lecture 16. The statements about dilations are proved similarly. □

Integral forumlas

Lemma

Suppose that $f \in \mathcal{L}^1(\mathbf{R})$ and $s \in \mathbf{R} \setminus \{0\}$. Then

$$\int_{-\infty}^{\infty} f(r-s)\,dm(r) = \int_{-\infty}^{\infty} f(r)\,dm(r) \quad \textit{and}$$
$$\int_{-\infty}^{\infty} f(sr)\,dm(r) = |s| \int_{-\infty}^{\infty} f(r)\,dm(r).$$

Proof.

If $E \in \mathcal{L}$, then $m(E+y) = m(E)$ and

$$\int_{-\infty}^{\infty} \mathbb{1}_E(r-s)\,dm(r) = \int_{-\infty}^{\infty} \mathbb{1}_{E+s}(r)\,dm(r) = \int_{-\infty}^{\infty} \mathbb{1}_E(r)\,dm(r).$$

Hence the first formula holds if f is a simple function. But there are simple functions $f_n \to f$ pointwise with $|f_n| \leq |f| \in \mathcal{L}^1(\mathbf{R})$. Hence the equation holds for all $f \in \mathcal{L}^1(\mathbf{R})$ by the LDCT. The second equation is proved similarly. □

$\mathbf{R}^2$

Definition

Let $(\mathbf{R}, \mathcal{L}, m)$ be Lebesgue measure on the real line. Then the completion $(\mathbf{R} \times \mathbf{R}, \mathcal{L}^2, m^2)$ of $(\mathbf{R} \times \mathbf{R}, \mathcal{L} \otimes \mathcal{L}, m \times m)$ is called Lebesgue measure on $\mathbf{R}^2$.

Remark

In a different course, we would introduce Lebesgue measure on $\mathbf{R}^n$, but there are some technicalities that I'd sooner avoid here as we bring this course to a close. But we should at least observe that since $\mathcal{B}(\mathbf{R} \times \mathbf{R}) = \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$ (by HW#46), $\mathcal{B}(\mathbf{R} \times \mathbf{R}) \subset \mathcal{L} \otimes \mathcal{L} \subset \mathcal{L}^2$, so continuous functions and even Borel functions on $\mathbf{R}^2$ are $\mathcal{L}^2$-measurable. But some subtleties remain.

Composition

Lemma

Suppose that $f : \mathbf{R} \to \mathbf{C}$ *is* $\mathcal{L}$*-measurable. Then*

$$k(r, s) = f(s - r)$$

is $\mathcal{L}^2$*-measurable on* $\mathbf{R}^2$*.*

Remark (Not Obvious)

If we had been reasonable—and started with a Borel function $f : \mathbf{R} \to \mathbf{C}$*—then the composition of* f *with a continuous function* $g : \mathbf{R}^2 \to \mathbf{R}$ *such as* $g(r, s) = s - r$ *would be Borel, and hence* $\mathcal{L}^2$*-measurable. But as* f *is only* $\mathcal{L}$*-measurable, the composition of* f *with even a continuous function need not be measurable. (I gave such an example in the optional write up of the Cantor-Lebesgue function.) However, we've established that the composition of a Borel function with a* $\mathcal{L}$*-measurable function, such as* f*, is measurable.*

Proof

Proof.

Let g be a Borel function such that $g = f$ almost everywhere. Then there is a Borel null set N such that $f(r) = g(r)$ if $r \notin N$. Let $k'(r,s) = g(s-r)$. Then k' is Borel and hence $\mathcal{L}^2$-measurable. Furthermore, $k(r,s) = k'(r,s)$ provided

$$(r,s) \notin D = \{\,(r,s) : s - r \in N\,\}.$$

Since m^2 is complete, it will suffice to see that D is a m^2-null set. But our Tonelli Theorem for complete measures implies

$$m^2(D) = \int_{\mathbf{R}^2} \mathbb{1}_D(r,s)\,dm^2(r,s) = \int_{-\infty}^{\infty} m(D_r)\,dm(r).$$

Since $D_r = \{\,s : s - r \in N\,\} = N + r$. $m(D_r) = 0$ for all r. □

Convolution

Definition

Suppose that f and g are Lebesgue measurable. Then their convolution $f * g$ is defined at $s \in \mathbf{R}$ whenever $r \mapsto f(r)g(s-r)$ is integrable, and then

$$f * g(s) = \int_{-\infty}^{\infty} f(r)g(s-r)\,dm(r).$$

If the convolution is defined almost everywhere, then we view $f * g$ as a function on all of $\mathbf{R}$ by defining $f * g(s) = 0$ if $r \mapsto f(r)g(s-r)$ is not integrable.

A Product on $L^1(\mathbf{R})$

Theorem

If $f, g \in \mathcal{L}^1(\mathbf{R})$, then $f * g$ is defined almost everywhere and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

In particular, the class of $f * g$ in $L^1(\mathbf{R})$ depends only on the classes of f and g. Moreover, convolution induces a commutative and associative product on $L^1(\mathbf{R})$: $[f] * [g] := [f * g]$.

Proof.

Since $(r, s) \mapsto f(r)g(s - r)$ is $\mathcal{L}^2$-measurable, we can apply Tonelli's Theorem for Complete Measures. Thus there is a null set N such that

$$s \mapsto |f(r)g(s - r)|$$

is measurable for all $r \notin N$.

Proof Continued.

Furthermore

$$\int_{\mathbf{R}^2} |f(r)g(s-r)|\,dm^2(r,s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(r)g(s-r)|\,dm(s)\,dm(r)$$

$$= \int_{-\infty}^{\infty} |f(r)| \int_{-\infty}^{\infty} |g(s-r)|\,dm(s)\,dm(r)$$

$$= \|f\|_1\|g\|_1 < \infty.$$

Therefore, $k(r,s) = f(r)g(s-r)$ is in $\mathcal{L}^1(\mathbf{R}^2)$. Now by Fubini's Theorem, $r \mapsto f(r)g(s-r)$ is integrable for almost all s! Thus $f * g$ is defined almost everywhere, and $\|f * g\|_1 \leq \|f\|_1\|g\|_1$.

Proof

Proof Continued.

Note that if $f \sim f'$, then $f * g - f' * g = (f - f') * g$. Since $\|(f - f') * g\|_1 = 0$, we have $f * g = f' * g$ in $L^1(\mathbf{R})$. Similarly, if $g \sim g'$, we have $[f * g] = [f * g']$. Hence we can view convolution as a binary operation on $L^1(\mathbf{R})$.

Also

$$f * g(s) = \int_{-\infty}^{\infty} f(r)g(s - r)\,dm(r) = \int_{-\infty}^{\infty} f(r + s)g(-r)\,dm(r)$$
$$= \int_{-\infty}^{\infty} f(-r + s)g(r)\,dm(r) = g * f(s).$$

Thus convolution is commutative.

Proof

Proof Continued.

Associativity requires Fubini. I'll just sketch the details.

$$\begin{aligned} f*(g*h)(s) &= \int_{-\infty}^{\infty} f(r)g*h(s-r)\,dr \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r)g(t)h(s-r-t)\,dt\,dr \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{f(r)g(t-r)h(s-t)}_{k_s(t,r)}\,dt\,dr \\ &\overset{\text{Fubini}}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r)g(t-r)h(s-t)\,dr\,dt \\ &= \int_{-\infty}^{\infty} f*g(t)h(s-t)\,dt \\ &= (f*g)*h(s). \end{aligned}$$

□

Break Time

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- Questions?
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Translation

1. If $f : \mathbf{R} \to \mathbf{C}$ is measurable, let $\lambda(s)f(r) = f(r - s)$.
2. Since $(\lambda(s)f)^{-1}(V) = f^{-1}(V) + s$, $\lambda(s)f : \mathbf{R} \to \mathbf{C}$ is measurable.
3. If $f \in \mathcal{L}^p(\mathbf{R})$, then $\|\lambda(s)f\|_p = \|f\|_p$.
4. Since $\lambda(s) : \mathcal{L}^p(\mathbf{R}) \to \mathcal{L}^p(\mathbf{R})$ is linear, it follows from (3) that $\lambda(s)$ extends to a linear isometry of $L^p(\mathbf{R})$ to itself.

Lemma

Suppose $1 \leq p < \infty$. For each $f \in \mathcal{L}^p(\mathbf{R})$, the map $s \mapsto \lambda(s)f$ is continuous from $\mathbf{R}$ to $L^p(\mathbf{R})$.

Sketch of the Proof.

If $f \in C_c(\mathbf{R})$, then f is uniformly continuous and the result is straightforward. Let $f \in \mathcal{L}^p(\mathbf{R})$, $r \in \mathbf{R}$, and fix $\epsilon > 0$. By HW, there is a $g \in C_c(\mathbf{R})$ such that $\|f - g\|_p < \epsilon/3$. Let $\delta > 0$ be such that $|s - r| < \delta$ implies $\|\lambda(s)g - \lambda(r)g\|_p < \epsilon/3$. Then if $|s - r| < \delta$, we have

$$\begin{aligned}
\|\lambda(s)f - \lambda(r)f\|_p &\leq \|\lambda(s)f - \lambda(s)g\|_p + \|\lambda(s)g - \lambda(r)g\|_p \\
&\qquad + \|\lambda(r)g - \lambda(r)f\|_p \\
&= \|f - g\|_p + \|\lambda(s)g - \lambda(r)g\|_p + \|g - f\|_p \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

□

Notation

If $g : \mathbf{R} \to \mathbf{C}$ *is a function, then we let* $\tilde{g}(r) = g(-r)$*. Then*

$$f * g(s) = \int_{-\infty}^{\infty} f(r) g(s-r)\, dm(r)$$
$$= \int_{-\infty}^{\infty} f(r)\lambda(s)\tilde{g}(r)\, dm(r). \qquad (1)$$

Lemma

Suppose $\frac{1}{p} + \frac{1}{q} = 1$*. If* $f \in \mathcal{L}^p(\mathbf{R})$ *and* $g \in \mathcal{L}^q(\mathbf{R})$*, then* $f * g$ *is defined everywhere and* $f * g : \mathbf{R} \to \mathbf{C}$ *is continuous.*

Proof.

Since $r \mapsto f(r)$ is in $\mathcal{L}^p(\mathbf{R})$ while $r \mapsto \lambda(s)\tilde{g}(r)$ is in $\mathcal{L}^q(\mathbf{R})$ for each $s \in \mathbf{R}$, $r \mapsto f(r)\lambda(s)\tilde{g}(r)$ is in $\mathcal{L}^1(\mathbf{R})$ for each s by Hölder. Therefore $f * g(s)$ is always defined in view of (1).

Proof

Proof Continued.

Since $f * g = g * f$, we can assume $q \neq \infty$. Again using (1).

$$f * g(s) - f * g(r) = \int_{-\infty}^{\infty} f(t)\big(\lambda(s)\tilde{g}(t) - \lambda(r)\tilde{g}(t)\big)\, dm(t)$$
$$\leq \|f\|_p \|\lambda(s)\tilde{g} - \lambda(r)\tilde{g}\|_q.$$

Now the result follows from the continuity of $r \mapsto \lambda(r)\tilde{g}$. □

A Fun Corollary

Lemma

Suppose $E \subset \mathbf{R}$ is such that $m(E) > 0$. Then

$$E - E = \{ x - y : x, y \in E \}$$

contains an open interval about 0.

Proof.

We can assume $0 < m(E) < \infty$. Hence $\mathbb{1}_E \in \mathcal{L}^1(\mathbf{R})$ and $\mathbb{1}_{-E} \in \mathcal{L}^\infty(\mathbf{R})$. Thus $f(s) = \mathbb{1}_E * \mathbb{1}_{-E}(s)$ is continuous. But

$$f(s) = \int_{-\infty}^{\infty} \mathbb{1}_E(r)\mathbb{1}_E(r - s)\, dm(r).$$

Hence $f(0) = m(E) > 0$ and $f(s) = 0$ if $s \notin E - E$. $\square$

That's Enough for Today

- That is enough for now.
- In fact, that is really enough for Math 73/103.
- We will talk about a different approach to defining measures—using linear functionals—on Monday.
- Monday's lecture "will not be on the exam".