Math 73/103: Fall 2020
Lecture 28

Dana P. Williams

Dartmouth College

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Getting Started

- We should be recording!
- Questions?
- Our final homework assignment is due tomorrow.
- I hope to send the final out by the end of the week.
- Today's lecture is meant as peak beyond the standard fare. It is not officially part of the course. It is "not on the exam", but it is the way measure theory arises in my work.

Another Approach

- In this course, we were only able to build interesting measures—such as Lebesgue measure on the real line or product measures—by extracting them from outer measures using "Carathéodory's Theorem".
- Another approach comes from the observation that if μ is a finite Borel measure—that is a measure defined on all Borel sets so that all continuous functions are μ-measurable—on a compact metric space X, then

$$\Lambda(f) = \int_X f(x)\,d\mu(x) \tag{1}$$

  is a linear functional on the complex vector space $C(X)$.
- Note that Λ is an additional property: it is positive is the sense that if $f \geq 0$ in $C(X)$, then $\Lambda(f) \geq 0$.
- As it turns out, we can reverse the process and construct a measure μ from a positive linear functional Λ such that (1) holds.

Some Topology

- To stay grounded, we will restrict to a separable metric space X (or equivalently a second countable metric space).
- If $f \in C(X)$, then the support of f is $\text{supp}(f) = \overline{\{ x : f(x) \neq 0 \}}$.
- We let $C_c(X)$ be the subspace of $C(X)$ of continuous functions with compact support.
- In order that $C_c(X)$ be rich, we need to suppose in addition that X is locally compact in that every point $x \in X$ has a neighborhood V such that $\overline{V}$ is compact.
- Therefore throughout this discussion, X will be a second countable locally compact Hausdorff space. (Such spaces always admit a metric generating the given topology.)
- For example, X could be either an open or closed subset of $\mathbf{R}^n$ or any manifold for that matter.
- It follows that X is σ-compact; that is, X is the increasing union of compact sets.

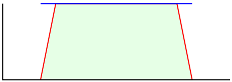
Functions in $C_c(X)$

- If $K \subset X$ is compact, then we write $K \prec f$ to denote a function $f \in C_c(X)$ such that for all x $0 \le f(x) \le 1$ and $f(x) = 1$ for all $x \in K$.
- Similarly, if $V \subset X$ is open, then we write $f \prec V$ to denote a function $f \in \mathbf{C}_c(X)$ such that $0 \le f(x) \le 1$ and $\operatorname{supp}(f) \subset V$.
- Suppose that $K \subset X$ is compact and V is an open neighborhood of K. Then K and $X \setminus V$ are disjoint closed sets. Then, since X is a metric space, in HW#10 we produced a continuous function f such that $0 \le f(x) \le 1$, $f(x) = 1$ on K and $f(x) = 0$ on $X \setminus V$. Since X is locally compact, we can arrange that $f \in C_c(X)$ and hence $K \prec f \prec V$. This result is often called Urysohn's Lemma.

Getting an Outer Measure

- If Λ is a positive linear functional on $C_c(X)$, we want to build a measure such that $\Lambda(f) = \int_X f(x)\,d\mu(x)$ for $f \in C_c(X)$.
- If $V \subset X$ is open, we define

$$\mu(V) = \sup\{\,\Lambda(f) : f \prec V\,\}.$$



- Since μ is clearly monotonic on open sets, we can extend μ to arbitrary subsets $E \subset X$ by

$$\mu(E) = \inf\{\,\mu(V) : E \subset V \text{ and } V \text{ is open}\,\}.$$

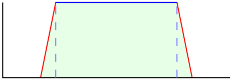
- It takes some work to see that μ is an outer measure on X.

Getting $\mathcal{M}$

- Instead of using Carathéodory's characterization of measurable sets, we let $\mathcal{M}_F$ be the collection of $E \subset X$ such that $\mu(E) < \infty$ and
$$\mu(E) = \sup\{\, \mu(K) : K \subset E \text{ and } K \text{ is compact} \,\}.$$
- Then we let $\mathcal{M}$ be the set of $E \subset X$ such that $E \cap K \in \mathcal{M}_F$ for all $K \subset X$ compact.

Getting a Measure

- If $K \subset X$ is compact, then $\mu(K) = \inf\{\, \Lambda(f) : K \prec f \,\}$.



In particular, $\mu(K) < \infty$ for all compact subsets $K \subset X$.

- There is hard work to do to show that $\mathcal{M}$ is a σ-algebra containing all closed sets—and hence all Borel sets—and that μ restricted to $\mathcal{M}$ is a complete measure.
- Then one can prove that for all $f \in C_c(X)$, we have

$$\Lambda(f) = \int_X f(x)\, d\mu(x).$$

Rieze Representation Theorem

Theorem (Rieze Representation Theorem)

Suppose that X is a second countable locally compact Hausdorff space, and let $\Lambda : C_c(X) \to \mathbf{C}$ be a positive linear functional. Then there is a complete σ-finite measure μ on a σ-algebra $\mathcal{M}$ containing $\mathcal{B}(X)$ such that

$$\Lambda(f) = \int_X f(x)\, d\mu(x) \quad \text{for all } f \in C_c(X). \tag{2}$$

Furthermore,

1. *$\mu(K) < \infty$ for all $K \subset X$ compact.*
2. *μ is outer regular in that*
$$\mu(E) = \inf\{\, \mu(V) : E \subset V \text{ and } V \text{ is open} \,\} \quad \text{for all } E \in \mathcal{M}.$$
3. *μ is inner regular in that*
$$\mu(E) = \sup\{\, \mu(K) : K \subset E \text{ and } K \text{ is compact} \,\} \quad \text{for all } E \in \mathcal{M}.$$
4. *If ν is any other measure defined on $\mathcal{B}(X)$ such that (2) holds, then $\nu(E) = \mu(E)$ for all $E \in \mathcal{B}(X)$.*

Back to our Course

Remark

Since the measure μ coming from the RRT is σ-finite and outer regular, we can apply HW#37 to conclude that if $E \in \mathcal{M}$ and $\epsilon > 0$ there is an open set V and a closed set F such that $F \subset E \subset V$ and $\mu(V \setminus F) < \epsilon$. Furthermore there is a G_δ-set G and a F_σ-set A such that $A \subset E \subset G$ with $\mu(G \setminus A) = 0$. It follows that $(X, \mathcal{M}, \mu)$ is the completion of $(X, \mathcal{B}(X), \mu)$.

Example

Note that the good old Riemann integral gives us a positive linear functional on $C_c(\mathbf{R})$: $\Lambda(f) = \int_a^b f$ provided $\operatorname{supp}(f) \subset [a, b]$. (It is not hard to check that this is well-defined.) Let $(\mathbf{R}, \mathcal{M}, \mu)$ be the measure space coming from the RRT. Since we proved our Lebesgue integral extends the Riemann integral, the uniqueness statement in the RRT implies that $(\mathbf{R}, \mathcal{M}, \mu) = (\mathbf{R}, \mathcal{L}, m)$. Note that the μ-integral extends the Riemann integral (almost) by definition!

Similarly, we recover Lebesgue measure on $\mathbf{R}^n$ via a functional build from the Riemann integral in n-dimensions.

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

$C_0(X)$

- Of course $C_c(X)$ is a normed vector space: $\|f\| = \sup_{x\in X} |f(x)|$.
- However, a positive linear functional $\Lambda : C_c(X) \to \mathbf{C}$ need not be bounded. The Riemann/Lebesgue integral is an example.
- We can complete $C_c(X)$ by taking its closure $C_0(X)$ in $C_b(X)$.
- I'll leave it as an exercise to check that $f \in C_0(X)$ if $\{ x : |f(x)| \geq \epsilon \}$ is compact for all $\epsilon > 0$. We say that such functions "vanish at infinity".

Bounded Functionals

- If μ is the measure associated to a positive linear functional $\Lambda : C_c(X) \to \mathbf{C}$ by the RRT, then
$$\mu(X) = \sup\{\, \Lambda(f) : f \in C_c(X) \text{ with } 0 \le f \le 1 \,\}.$$
- It follows that we can extend Λ to a bounded (positive) functional on $C_0(X)$ if and only if $\mu(X) < \infty$.
- Thus if Λ is bounded, then $\|\Lambda\| = \mu(X)$.
- In fact, any positive linear functional on $C_0(X)$ is bounded, but the proof is subtle. It turns out that $\mu(X) = \infty$ implies there is a $f \in C_0(X)$ such that $f \ge 0$ and $\int_X f \, d\mu = \infty$.

Getting There

- A measure on X is called a Borel measure if it is defined on (at least) the Borel σ-algebra $\mathcal{B}(X)$.
- Since continuous functions are μ-measurable for any Borel measure μ, it follows from the RRT that positive linear functionals on $C_0(X)$ are in one-to-one correspondence with finite Borel measures on X: $\mu \mapsto \Lambda_\mu$ where

$$\Lambda_\mu(f) = \int_X f(x)\,d\mu(x) \quad \text{and} \quad \|\Lambda_\mu\| = \mu(X).$$

- Recall that if ν is a complex measure on $(X, \mathcal{M})$, then there are finite (positive) measures μ_k such that $\nu(E) = \mu_1(E) - \mu_2(E) + i\mu_3(E) - i\mu_4(E)$.
- Therefore each complex Borel measure ν on $\mathcal{B}(X)$ determines $\Lambda_\nu \in C_0(X)^*$ by

$$\Lambda_\nu(f) = \Lambda_{\mu_1}(f) - \Lambda_{\mu_2}(f) + i\Lambda_{\mu_3}(f) - i\Lambda_{\mu_4}(f).$$

Total Variation

- We let $M(X)$ be the collection of all complex Borel measures on X.
- If $\nu \in M(X)$, then there is a unique finite measure $|\nu|$ and a measurable function $\varphi : X \to \{ z \in \mathbf{C} : |z| = 1 \}$ determined $|\nu|$-almost everywhere such that

$$\nu(E) = \int_E \varphi(x)\, d|\nu|(x).$$

- For example, if ν is real-valued with Jordan decomposition $\nu^+ - \nu^-$ then $|\nu| = \nu^+ + \nu^-$ with $\varphi \equiv 1$ on the support of ν^+ and $\varphi \equiv -1$ on the support of ν^-.
- We call $|\nu|$ the total variation of ν.
- $M(X)$ is a complex vector space and $\|\nu\| = |\nu|(X)$ is a complete norm on $M(X)$ such that $\|\Lambda_\nu\| = \|\nu\|$.

The Dual of $C_0(X)$

Theorem

Suppose that X is a second countable locally compact Hausdorff space. Then the map $\nu \mapsto \Lambda_\nu$ is an isometric isomorphism of $M(X)$ onto $C_0(X)^*$.

Proof.

The crux of the proof is a sort of Jordan decomposition for bounded linear functionals $\Lambda : C_0(X) \to \mathbf{R}$ as a difference $\Lambda^+ - \Lambda^-$ of positive linear functionals to which the RRT applies. □

Comments

Remark

The RRT is holds with some minor caveats without assuming X is second countable or even metrizable. My favorite source for the RRT is Chapter 2 of Rudin's Real & Complex. Folland's treatment in §7.1–3 of his Real Analysis is also very good, and also includes the application to $C_0(X)^*$. All this material is also covered in Chapter 21 of Royden & Fitzpatrick.

That's Enough

- That is enough.
- Remember that today was "just for fun", and we completed the course on Friday.
- Your last homework is due tomorrow.
- I hope to email out the final in a few days. It is due on the 30th.