Math 73/103: Fall 2020
Lecture 3

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Getting Started

- We should be recording!
- This a good time to ask questions about the previous lecture, complain, or tell a story.
- As I mentioned, I hope that you have the bandwidth to keep your video on during the class meeting. This makes it seem a little more "real" for me. But this is voluntary.
- `Gradescope` may not work for us. My guess is that you will be uploading your PDFs from canvas. More next week: homework problems 1–10 will be due Wednesday in any case.

A Few Repairs

Contrary to what I said in lecture one, the triangle inequality for a metric ρ does not imply $\rho(x,x) = 0$ for all x. This needs to be part of the axioms!

Definition (Corrected)

A metric on a set X is a function $\rho : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ we have

1. [definiteness] $\rho(x, y) = 0$ if and only if $x = y$,
2. [symmetry] $\rho(x, y) = \rho(y, x)$ and
3. [triangle inequality] $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

Then we call ρ a pusedo metric if item one is replaced by merely $\rho(x, x) = 0$ for all x.

Reverse Triangle Inequality

Lemma

Suppose that (X,ρ) is a metric space. Then for all $x,y,z\in X$, we have

$$\big|\rho(x,z)-\rho(x,y)\big|\le\rho(z,y).$$

Proof.

By the triangle inequality, we have $\rho(x,z)\le\rho(x,y)+\rho(y,z)$. that is

$$\rho(x,z)-\rho(x,y)\le\rho(y,z)=\rho(z,y).$$

By symmetry,

$$\rho(x,y)-\rho(x,z)\le\rho(z,y).$$

The result follows. □

Nested Sets

Definition

A metric space has the nested set property if given a sequence (F_n) of closed nonempty sets F_n such that $F_{n+1} \subset F_n$ for all n and such that $\operatorname{diam}(F_n) \to 0$, then there is a unique $x \in X$ such that

$$\bigcap_{n=1}^{\infty} F_n = \{ x \}.$$

Theorem

A metric space X is complete if and only if it has the nested set property.

Remark

Note that the word "nonempty" in the definition of the nested set property is critical.

Proof

Proof.

Suppose that X is complete and (F_n) are as in the statement of the theorem. Pick $x_n \in F_n$. Then if $m \geq n \geq M$, we have $x_n, x_m \in F_M$ and

$$\rho(x_n, x_m) \leq \operatorname{diam}(F_M).$$

Since $\operatorname{diam}(F_M) \to 0$, it follows that (x_n) is Cauchy. Therefore there is a $x \in X$ such that $x_n \to x$. Since $x = \lim_{n \geq M} x_n$, we have $x \in F_M$ for all $M \geq 1$. Hence $x \in \bigcap F_n$. But if $y \in \bigcap F_n$, then $\rho(x, y) \leq \operatorname{diam}(F_n)$ for all n. Hence $\rho(x, y) = 0$ and $x = y$. Thus X has the nested set property.

Proof

Proof Continued.

Now suppose that X has the nested set property. Let (x_n) be a Cauchy sequence in X. It will suffice to prove that (x_n) converges. Let $F_n = \overline{\{ x_k : k \geq n \}}$. Clearly, $F_{n+1} \subset F_n$ (because $A \subset B$ implies $\overline{A} \subset \overline{B}$).

Let M be such that $n, m \geq M$ implies $\rho(x_n, x_m) < \epsilon$. Let $A = \{ x_k : k \geq M \}$ so that $F_M = \overline{A}$. Let $x, y \in F_M$ and fix $\delta > 0$. Then there are $a, b \in A$ such that $\rho(x, a) < \delta/2$ and $\rho(y, b) < \delta/2$. Then using the reverse triangle inequality,

$$\begin{aligned}
\rho(x, y) &\leq |\rho(x, y) - \rho(a, b)| + \rho(a, b) \\
&\leq |\rho(x, y) - \rho(x, b)| + |\rho(x, b) - \rho(a, b)| + \rho(a, b) \\
&\leq \rho(y, b) + \rho(x, a) + \rho(a, b) < \delta + \epsilon.
\end{aligned}$$

Since δ is arbitrary, $\rho(x, y) \leq \epsilon$. Since $x, y \in F_M$ are arbitrary, $\mathrm{diam}(F_M) \leq \epsilon$.

Proof

Proof Continued.

We have shown that $\operatorname{diam}(F_n) \to 0$. Hence

$$\bigcap F_n = \{ x \}$$

for some $x \in X$. Since $\rho(x_n, x) \leq \operatorname{diam}(F_n)$, it follows that $x_n \to x$. Therefore X is complete. □

- Time for a break.
- Are there questions or comments?
- Start recording again.

Compactness

Definition

If $E \subset X$, then a family $\mathcal{U} = \{ U_i \}_{i \in I} \subset \mathcal{P}(X)$ is called a cover of E if

$$E \subset \bigcup_{i \in I} U_i.$$

A subset $\mathcal{V} \subset \mathcal{U}$—say $\mathcal{V} = \{ U_j \}_{j \in J}$ with $J \subset I$— is called a subcover of $\mathcal{U}$ if $\mathcal{V}$ also covers E. That is,

$$E \subset \bigcup_{j \in J} U_j.$$

If X is a metric space, then a cover $\mathcal{U} = \{ U_i \}_{i \in I}$ is called open if each U_i is open in X.

The Defintion

Definition

A metric space X is compact if every open over of X has a finite subcover.

Remark

Never has a definition seemed—at first blush—to be so odd and unmotivated. Full disclosure—this version of the definition did not come first. Nevertheless, the open cover concept has proven itself to be very valuable.

This Definition is Subtle

Example

Consider $X = [0, 1) \subset \mathbf{R}$ (with the usual metric). Of course, X has finite open covers. Every metric space does. Let $\mathcal{V}$ be the open cover consisting of the sets $[0, x)$ and $(y, 1)$ for all $0 < x, y < 1$. then $\{ [0, \frac{3}{4}), (\frac{1}{2}, 1) \}$ is a finite subcover. This proves nothing! But $\mathcal{U} = \{ [0, x) : 0 < x < 1 \}$ is an open cover which has no finite subcover:

$$\bigcup_{k=1}^{n} [0, x_k) = [0, x_{k_0}) \subsetneq [0, 1)$$

where $x_{k_0} = \max_{1 \le k \le n} x_k$. This shows that $[0, 1)$ is not compact.

Compactness

Example

Any finite metric space is compact.

Proposition

Suppose that K is a subspace of a metric space X. Then K is compact if and only if every open over of K in X has a finite subcover.

Proof.

Suppose that K is compact. Let $\{ U_i \}_{i\in I}$ be an open cover of K in X. That is, $K \subset \bigcup_{i\in I} U_i$, and each U_i is open in X. Then $U_i \cap K$ is open in K and $K = \bigcup_{i\in I} U_i \cap K$. Since K is compact, there are $i_i, \dots, i_n$ such that $K = \bigcup_{j=1}^{n} U_{i_j} \cap K$. Then $K \subset \bigcup_{j=1}^{n} U_{i_j}$.

The Converse

Proof Continued.

Now suppose that K has the given property in X. Let $K = \bigcup_{i \in I} v_i$ be an open over of K (in K). Then there are open sets U_i in X such that $V_i = U_i \cap K$. Then $K \subset \bigcup_{i \in I} U_i$. By assumption, there are $i_1, \ldots, i_n$ such that $K \subset \bigcup_{j=1}^{n} U_j$. Then $K = \bigcup_{j=1}^{n} V_j$. Hence K is compact as required. □

Closed Sets

Definition

A collection $\mathcal{F}$ of subsets of X has the finite intersection property (FIP) if given $F_1, \dots, F_n \in \mathcal{F}$, we have $\bigcap_{k=1}^n F_k \neq \emptyset$.

Example

Let $F_n = [n, \infty) \subset \mathbf{R}$ and $\mathcal{F} = \{ F_n \}_{n \in \mathbf{N}}$. Then $\mathcal{F}$ is a collection of closed subsets of $\mathbf{R}$ with the FIP.

Proposition

A metric space X is compact if and only if every collection $\mathcal{F}$ of closed subsets of X with the FIP satisfies $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$.

Proof.

Take complements. □

Example

$\mathbf{R}$ is not compact: $\bigcap_n [n, \infty) = \emptyset$.

Total Boundedness

Remark

Recall that a subset of a metric space is bounded if it has finite diameter. It is not hard to see that B is bounded if and only if $B \subset B_r(x_0)$ for some $x_0 \in X$ and $r > 0$.

Definition

A metric space X is totally bounded if for all $\epsilon > 0$, X has a finite cover by ϵ-balls. A subspace $E \subset X$ is totally bounded, if it is totally bounded as a subspace.

Definition

Suppose that E is a subspace of X and that $\epsilon > 0$. Then an ϵ-net for E is a finite collection $\{ B_\epsilon(x_i) \}_{i=1}^n$ with $x_i \in X$ such that $E \subset \bigcup_{i=1}^n B_\epsilon(x_i)$.

The Point

Lemma

Subset E in a metric space X is totally bounded if and only if there is an ϵ-net for E in X for all $\epsilon > 0$.

Proof.

I am leaving this for homework. The issue is that if $\{ B_\epsilon(x_i \}_{i=1}^n$ is an ϵ-net for E in X, then $\{ B_\epsilon^E(x_i) \}_{i=1}^n$ is not a cover of E by ϵ-balls in E since we don't necessarily have $x_i \in E$! But this is easily fixed. □

What Are We On About?

Example

Let $X = \ell^2$ and let $B = \{ x \in \ell^2 : \|x\|_2 \leq 1 \}$ be the closed unit ball. Clearly, B is bounded: $\operatorname{diam}(B) = 2$. Let $e_n \in \ell^2$ be given by

$$e_n(k) = \begin{cases} 1 & \text{if } k = n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

(More succinctly, $e_n(k) = \delta_{nk}$.) Then $e_n \in B$ and if $n \neq m$, then $\|e_n - e_m\|_2 = \sqrt{2}$. Let $\epsilon = \sqrt{2}/3$. Then any ϵ-ball in ℓ^2 can contain at most one e_n! Therefore, there is no ϵ-net for B in ℓ^2. In short, B is bounded by not totally bounded.

Why Our Intuition is Wrong

Proposition

A subset of $(\mathbf{R}^n, \|\cdot\|_2)$ *is bounded if and only if it is totally bounded.*

Remark

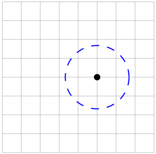
The same is true for any metric strongly equivalent to that induced by $\|\cdot\|_2$.

Proof.

Since totally bounded sets are easily seen to be bounded in any metric space, it suffices to consider a bounded subspace $E \subset \mathbf{R}^n$ and show that it is totally bounded. Hence given $\epsilon > 0$, it suffices to find an ϵ-net for E.

Since E is bounded, there is a $a > 0$ such that

$$E \subset [-a, a] \times \cdots \times [-a, a] = [-a, a]^n.$$



Proof Continued.

Let P_k be a regular partition of $[-a, a]$ such that the length of each subinterval is bounded by $\frac{1}{k}$. Then $P_k \times \cdots \times P_k$ gives a partition of $[-a, a]^n$ into n-cubes of diameter bounded by $\sqrt{n}/k$. If we let k be such that $\sqrt{n}/k < \epsilon$. This means that if $x \in P_k^n$, then $B_\epsilon(x)$ contains every n-cube for which x is a vertex. Hence $E \subset \bigcup_{x \in P_k^n} B_\epsilon(x)$. □

Definition

A metric space X is sequentially compact if every sequence in X has a convergent subsequence in X.

Example

Let $X = \{ \frac{1}{n} : n \in \mathbf{N} \} \cup \{ 0 \}$. I claim that X is sequentially compact.

Proof.

Let (x_k) be a sequence in X. Suppose that for some n, $\{ k : x_k = \frac{1}{n} \}$ is infinite. Then we can find k_1 such that $x_{k_1} = \frac{1}{n}$. If we have found $k_1 < k_2 < \cdots k_r$ such that $x_{k_j} = \frac{1}{n}$ for all $1 \leq j \leq r$, then we can find $k_{r+1} > k_r$ such that $x_{k_{r+1}} = \frac{1}{n}$. That is we can find a constant subsequence (x_{k_j}) converging to $\frac{1}{n} \in X$. Otherwise, we can assume $\{ k : x_k = \frac{1}{n} \}$ is finite for all n. Then I claim (x_k) already converges to $0 \in X$. Let $\epsilon > 0$. Then there is a N such that $\frac{1}{N} < \epsilon$. Note that there are at most finitely many k such that $x_k = \frac{1}{n}$ with $n \leq N$. Hence there is a M such that $k \geq M$ implies x_k is either 0 or is of the form $\frac{1}{n}$ with $n \geq N$. Therefore $k \geq M$ implies $|x_k - 0| < \epsilon$ as required. □

Characterizing Compact Metric Spaces

- Time for another break.
- First, questions and comments.
- Start recording again.

Let's see if there is time to prove a big theorem.

Theorem

Let X be a metric space. The following are equivalent.

1. *X is compact.*
2. *X is complete and totally bounded.*
3. *X is sequentially compact.*

Proof

(2)$\Longrightarrow$(1).

We suppose that X is complete and totally bounded. To the contrary of what we want to show, assume that X is not compact. Then X has an open over $\mathcal{U} = \{ U_i \}_{i \in I}$ which has no finite subcover.

Since X is totally bounded, it has a finte cover by $\frac{1}{2}$-balls. At least one of these can't be covered by finitely many U_i. Let F_1 be its closure. Note that $\operatorname{diam}(F_1) \leq 1$ and F_1 can't be covered by finitely many U_i.

Now over X by finitely many $\frac{1}{4}$-balls. At least one of these has a nonempty intersection with F_1 that can't be covered by finitely many U_i. Let F_2 be the closure of the intersection of that $\frac{1}{4}$ ball with F_1. Note that $F_2 \subset F_1$ and that $\operatorname{diam}(F_2) \leq \frac{1}{2}$. Furthermore, F_2 can't be covered by finitely many U_i.

Proof

(2)$\Longrightarrow$(1) continued.

Continuing inductively, we nonempty closed sets F_n such that $F_{n+1} \subset F_n$, $\text{diam}(F_n) \leq \frac{1}{n}$, and no F_n can be covered by finitely many U_i.

Since X is complete, it has the nested set property and there is a unique $x \in X$ such that $\{ x \} = \bigcap F_n$.

But there is an i_0 such that $x \in U_{i_0}$. Since U_{i_0} is open, there is a $r > 0$ such that $B_r(x) \subset U_{i_0}$. Since $\text{diam}(F_n) \to 0$, there is a n such that

$$x \in F_n \subset B_r(x) \subset U_{i_0}.$$

But this is a contradiction.

Thus we have proved (2)$\Longrightarrow$(1).

That's Enough for Today

- That is enough for now.