Math 73/103: Fall 2020
Lecture 4

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Getting Started

- We should be recording!
- This a good time to ask questions about the previous lecture, complain, or tell a story.
- As I mentioned, I hope that you have the bandwidth to keep your video on during the class meeting. This makes it seem a little more "real" for me. But this is voluntary.
- I am still fighting with gradescope. Probably you will be asked to upload your assignment Wednesday to canvas instead. But let's wait until lecture Wednesday.

Last Time

Theorem

Let X be a metric space. The following are equivalent.

1. *X is compact.*
2. *X is complete and totally bounded.*
3. *X is sequentially compact.*

(2)$\Longrightarrow$(1).

We suppose that X is complete and totally bounded. To the contrary of what we want to show, assume that X is not compact. Then X has an open over $\mathcal{U} = \{ U_i \}_{i \in I}$ which has no finite subcover.

Since X is totally bounded, it has a finte cover by $\frac{1}{2}$-balls. At least one of these can't be covered by finitely many U_i. Let F_1 be its closure. Note that $\operatorname{diam}(F_1) \leq 1$ and F_1 can't be covered by finitely many U_i.

Now over X by finitely many $\frac{1}{4}$-balls. At least one of these has a nonempty intersection with F_1 that can't be covered by finitely many U_i. Let F_2 be the closure of the intersection of that $\frac{1}{4}$ ball with F_1. Note that $F_2 \subset F_1$ and that $\operatorname{diam}(F_2) \leq \frac{1}{2}$. Furthermore, F_2 can't be covered by finitely many U_i.

Proof

(2)$\Longrightarrow$(1) continued.

Continuing inductively, we nonempty closed sets F_n such that $F_{n+1} \subset F_n$, $\operatorname{diam}(F_n) \leq \frac{1}{n}$, and no F_n can be covered by finitely many U_i.

Since X is complete, it has the nested set property and there is a unique $x \in X$ such that $\{ x \} = \bigcap F_n$.

But there is an i_0 such that $x \in U_{i_0}$. Since U_{i_0} is open, there is a $r > 0$ such that $B_r(x) \subset U_{i_0}$. Since $\operatorname{diam}(F_n) \to 0$, there is a n such that

$$x \in F_n \subset B_r(x) \subset U_{i_0}.$$

But this is a contradiction.

Thus we have proved (2)$\Longrightarrow$(1).

Proof

(1)$\Longrightarrow$(3).

Now we assume that X is compact. Let (x_n) be a sequence in X. We need to see that (x_n) has a convergent subsequence.

Let $F_n = \overline{\{\, x_k : k \geq n \,\}}$. Clearly $\mathcal{F} = \{\, F_n : n \in \mathbf{N} \,\}$ has the FIP. Since X is compact, there is a $x \in \bigcap_n F_n$.

Since $x \in F_1 = \overline{\{\, x_k : k \geq 1 \,\}}$, we must have $B_1(x) \cap \{\, x_k : k \geq 1 \,\}$. Choose n_1 such that $x_{n_1} \in B_1(x)$.

Similarly, $x \in F_{n_1+1} = \overline{\{\, x_k : k \geq n_1 + 1 \,\}}$. Hence there is a $n_2 > n_1$ such that $x_{n_2} \in B_{\frac{1}{2}}(x)$.

Continuing inductively, we get a subsequence (x_{n_k}) such that $x_{n_k} \in B_{\frac{1}{k}}(x)$. Hence $x_{n_k} \to x$. This establishes that (1)$\Longrightarrow$(3).

Proof

(3)$\Longrightarrow$(2).

Now we assume X is sequentially compact. Let ρ be the metric on X. we are tasked with showing that X is *both* totally bounded and complete.

Suppose to the contrary that X is not totally bounded. Then there is a $\epsilon > 0$ so that X can't be covered by finitely many ϵ-balls.

Pick $x_1 \in X$. Then $X \setminus B_\epsilon(x_1)$ is nonempty and there is a $x_2 \in X$ such that $\rho(x_1, x_2) \geq \epsilon$.

But $X \setminus \big(B_\epsilon(x_1) \cup B_\epsilon(x_2)\big) \neq \emptyset$. Hence there is a $x_3 \in X$ such that $\rho(x_3, x_1) \geq \epsilon$ and $\rho(x_3, x_2) \geq \epsilon$.

Continuing inductively, we get a sequence (x_n) such that $\rho(x_n, x_m) \geq \epsilon$ if $n \neq m$. Of course, such a sequence can have no subsequence which is Cauchy let alone convergent.

Hence X must be totally bounded if X is sequentially compact.

Proof

(3)$\Longrightarrow$(2) continued.

Finally, if X is sequentially compact and (x_n) is a Cauchy sequence, then (x_n) must have a convergent subsequence. Hence (x_n) is convergent (by a homework problem). Hence X is complete as well. This completes the proof of (3)$\Longrightarrow$(2), and also completes the proof of the theorem. □

Some Observations

- Whether or not a subset K of a metric space (X, ρ) is compact depends only on the topology τ_ρ.
- Hence if ρ and σ are equivalent metrics on X, then (X, ρ) is compact (respectively, sequentially compact) if and only if (X, σ) is compact (resp., sequentially compact).

Remark

We know that $\mathbf{R}$ and $\mathbf{C}$ are complete with respect to their usual metrics. The same is true of $(\mathbf{R}^n, \|\cdot\|_p)$ and $(\mathbf{C}^n, \|\cdot\|_p)$ for any $1 \leq p \leq \infty$. For example, we can view $(\mathbf{R}^n, \|\cdot\|_p)$ as a closed subspace of $\ell^p_{\mathbf{R}}$. Or we could just observe that if (x_n) is Cauchy in ℓ^p, then $(x_n(k))$ is Cauchy in $\mathbf{C}$ for all k.

Corollary

A subspace $K \subset (\mathbf{R}^n, \|\cdot\|_2)$ is compact if and only if it is closed and bounded.

Proof.

Since $\mathbf{R}^n$ is complete, K is complete if and only if it is closed. We also proved that subsets of $\mathbf{R}^n$ are bounded if and only if they are totally bounded. □

Theorem (Bolzano-Weierstrass Theorem)

Every bounded sequence in $\mathbf{R}^n$ has a convergent subsequence.

Extreme Value Theorem

Theorem (Extreme Value Theorem)

Suppose that X is a compact metric space and $f \in C(X, \mathbf{R})$—that is, $f : X \to \mathbf{R}$ is continuous. Then f attains its maximum and minimum on X. (That is, there are $x_1, x_2 \in X$ such that $f(x_1) \le f(x) \le f(x_2)$ for all $x \in X$.) In particular, if $f \in C(X)$, then f is bounded.

Proof.

If $f : X \to \mathbf{C}$ is continuous, then so is $|f| : X \to \mathbf{R}$. Hence the second assertion follows from the first. But you will prove on homework that if $f \in C(X, \mathbf{R})$, then $f(X)$ is compact in $\mathbf{R}$. This means that $f(X)$ is closed and bounded. But if $M = \sup_{x \in X} f(x)$, then $M < \infty$ and there is a sequence $(x_n) \subset X$ such that $f(x_n) \to M$. Hence $M \in f(X)$ and there is a $x \in X$ such that $f(x) = M$.

For the minimum, replace f by $-f$. □

Why was "Attain" in Red?

Example

Let $f(t) = \dfrac{1}{1+t^2}$. Then $f : \mathbf{R} \to \mathbf{R}$ is continuous and never vanishes. However, there is no $m > 0$ such that $f(t) \geq m$ for all $t \in \mathbf{R}$.

Corollary

Suppose that X is a compact metric space and $f \in C(X)$ never vanishes on X. Then there is a $m > 0$ such that $|f(x)| \geq m$ for all $x \in X$.

Proof.

We can apply the Extreme Value Theorem to $g(x) = |f(x)|$. Then there is a $x_0 \in X$ such that

$$0 < g(x_0) = m \leq g(x) \quad \text{for all } x \in X. \qquad \square$$

Break Time

- Let's Take a Break.
- First, Questions?
- Restart recording.

Lebesgue Numbers

Definition

Let $\mathcal{U} = \{ U_i \}_{i \in I}$ be an open cover of a metric space (X, ρ). We say that $d > 0$ is a Lebesgue number for $\mathcal{U}$ if for all $x_0 \in X$ there is a $i_0 \in I$ such that $B_r(x_0) \subset U_{i_0}$.

Example

Let $X = \mathbf{R}$, $U_1 = (-\infty, 1)$, $U_2 = (0, 2)$, and $U_3 = (1, \infty)$. Then any $0 < d \leq \frac{1}{2}$ is a Lebesgue number for $\mathcal{U} = \{ U_1, U_2, U_3 \}$: if $r \geq \frac{3}{2}$, then $B_d(r) \subset (1, \infty)$. If $r \leq \frac{1}{2}$, then $B_d(r) \subset (-\infty, 1)$. Then if $r \in (\frac{1}{2}, \frac{3}{2})$, then $B_d(r) \subset (0, 2)$.

Example

Example

Since $f(x) = \frac{1}{x}$ is continuous on $(0,1)$, there is a $\delta_x > 0$ so that $y \in B_{\delta_x}(x) = \{\, y \in (0,1) : |y - x| < \delta_x \,\}$ implies

$$\left| \frac{1}{y} - \frac{1}{x} \right| < 1.$$

Then

$$(0,1) \subset \bigcup_{x \in (0,1)} B_{\delta_x}(x).$$

For a homework problem, you can verify that this cover does not have a Lebesgue number.

Covering Lemma

Theorem (Lebesgue Covering Lemma)

Every open cover of a compact space has a Lebesgue number.

Proof.

Suppose that (X, ρ) is compact and that $X = \bigcup_{i \in I} U_i$ does not have a Lebesgue number. Then for all $n \geq 1$, there is a $x_n \in X$ such that $B_{\frac{1}{n}}(x_n)$ is not contained in any U_i. But (x_n) must have a convergent subsequence $x_{n_k} \to x_0$ in X. Then there is an i_0 such that $x_0 \in U_{i_0}$. But there is a $r > 0$ such that $B_r(x_0) \subset U_{i_0}$. Pick k such that $\rho(x_{n_k}, x_0) < \frac{r}{2}$ and $\frac{1}{n_k} < \frac{r}{2}$.

Now if $y \in B_{1/n_k}(x_{n_k})$, then

$$\rho(y, x_0) \leq \rho(y, x_{n_k}) + \rho(x_{n_k}, x_0) < \frac{1}{n_k} + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r.$$

This shows that $B_{1/n_k}(x_{n_k}) \subset U_{i_0}$. But this contradicts our assumption on x_{n_k}. □

Uniform Continuity

Theorem

Suppose that (X, ρ) is compact and that $f : (X, \rho) \to (Y, \sigma)$ is continuous. Then f is uniformly continuous.

Proof.

Fix $\epsilon > 0$. We need to find $\delta > 0$ such that $\rho(x, y) < \delta$ implies $\sigma\big(f(x), f(y)\big) < \epsilon$. Since f is continuous, for all $z \in X$, there is a $\delta_z > 0$ so that $f\big(B_{\delta_z}(z)\big) \subset B_{\epsilon/2}\big(f(z)\big)$. Let δ be a Lesbesgue number for the cover $X = \bigcup_{z \in X} B_{\delta_z}(z)$. Now if $\rho(x, y) < \delta$, there there is a z such that $B_\delta(x) \subset B_{\delta_z}(z)$. But then if $y \in B_\delta(x)$, both x and y are in $B_{\delta_z}(z)$ and

$$\sigma\big(f(x), f(y)\big) \leq \sigma\big(f(x), f(z)\big) + \sigma\big(f(z), f(y)\big) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This is what we wanted to show. □

Break Time

- Let's Take a Break.
- First, Questions?
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Equicontinuity

Definition

Let (X, ρ) be a metric space and $C(X)$ the complex vector space of continuous functions on X. We say that $\mathcal{F} \subset C(X)$ is equicontinuous at $x_0 \in X$ if for all $\epsilon > 0$ there is a $\delta > 0$ so that for all $f \in \mathcal{F}$

$$f\big(B_\delta(x_0)\big) \subset B_\epsilon\big(f(x_0)\big).$$

We say that $\mathcal{F}$ is equicontinuous on X if $\mathcal{F}$ is equicontinuous at every $x_0 \in X$.

Remark

Note that if $\mathcal{F} = \{ f \}$, then $\mathcal{F}$ is equicontinuous whenever f is continuous. It is not hard to see that if $\mathcal{F} \subset C(X)$ is finite, then $\mathcal{F}$ is always equicontinuous.

Example

Example

Let $X = [0,1]$, and let $f_n(x) = x^n$ for all $n \geq 1$. Let $x_n = \left(\frac{1}{2}\right)^{\frac{1}{n}}$. Note that $x_n \nearrow 1$. Furthermore,

$$\left| f_n(x_n) - 1 \right| = \frac{1}{2}.$$

It follows that $\mathcal{F} = \{ f_n : n \geq 1 \}$ is not equicontinuous at 1.

Remark

As a challenging homework problem, you are to show that $\mathcal{F} = \{ f_n : n \geq 1 \}$ is equicontinuous at each $x \in [0,1)$.

Preliminaries

Definition

A sequence (f_n) of functions on X is uniformly bounded if there is a $M > 0$ such that $|f_n(x)| \le M$ for all $x \in X$ and all $n \ge 1$. We say that (f_n) is pointwise bounded if for each $x \in X$ there is a $M_x > 0$ such that $|f_n(x)| \le M_x$ for all $n \ge 1$.

Example

Let $f_n(x) = \begin{cases} |x| & \text{if } |x| \le n, \\ n & \text{if } |x| \ge n. \end{cases}$. Then (f_n) is pointwise bounded on $\mathbf{R}$; we can let $M_x = |x|$. But (f_n) is not uniformly bounded.

Dense Sets

Definition

A subset of D of metric space X is dense if $\overline{D} = X$.

Lemma

Let D be a subset of a metric space X. The following are equivalent.

1. *D is dense in X.*
2. *Given $x \in X$, there is a sequence (d_n) in D such that $d_n \to x$.*
3. *$U \cap D \neq \emptyset$ for any nonempty open set U in X.*

Proof.

(1)$\Longrightarrow$(2): If $\overline{D} = X$, then every point in X is a limit point of D.

(2)$\Longrightarrow$(3): Suppose that $x \in U$ and than U is open. Let $(d_n) \subset D$ be such that $d_n \to x$. Then (d_n) is eventually in U and $U \cap D \neq \emptyset$.

(3)$\Longrightarrow$(1): Suppose that $\overline{D} = F \subsetneq X$. Then $U = X \setminus F$ is nonempty and open. Hence $D \cap U \neq \emptyset$. □

Examples

Example

The rationals, $\mathbf{Q}$ are dense in $\mathbf{R}$ and $\mathbf{Q} + i\mathbf{Q}$ is dense in $\mathbf{C}$. More generally, $\mathbf{Q}^n$ and $(\mathbf{Q} + i\mathbf{Q})^n$ are dense in $\mathbf{R}^n$ and $\mathbf{C}^n$ with respect to any of the metrics induced by the p-norms for $1 \leq p \leq \infty$.

Example

Let D be the set of complex sequences that are eventually zero. Thus $d \in D$ if there is a $N \in \mathbf{N}$ such that $d(k) = 0$ if $k \geq N$. Then D is dense in ℓ^p for any $1 \leq p < \infty$.

Proof.

If $x \in \ell^p$, then $\sum_{k=1}^{\infty} |x_k|^p < \infty$. Hence given $\epsilon > 0$, there is a N such that $\sum_{k=N}^{\infty} |x_k|^p < \epsilon^p$. Thus if $y \in D$ is the the sequence x truncated at N, $\|x - y\|_p < \epsilon$. □

That's Enough for Today

- That is enough for now.