Math 73/103: Fall 2020
Lecture 5

Dana P. Williams

Dartmouth College

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Getting Started

- We should be recording!
- This a good time to ask questions about the previous lecture, complain, or tell a story.
- As I mentioned, I hope that you have the bandwidth to keep your video on during the class meeting. This makes it seem a little more "real" for me. But this is voluntary.
- I am still fighting with gradescope. For today's assignment, please upload it directly to canvas.

Separable Spaces

Definition

A metric space X is called separable if it has a countable dense subset.

Example

Note that $\mathbf{R}^n$ and $\mathbf{C}^n$ are separable. In fact, ℓ^p is separable for all $1 \leq p < \infty$. However, ℓ^∞ is not separable.

Proof.

To see that ℓ^p is separable, consider the set D' of sequences taking values in $\mathbf{Q} + i\mathbf{Q}$ and which are eventually zero. To see the ℓ^∞ can't be separable, notice that for each $A \subset N$, let x_A be the sequence such that $x_A(k) = 1$ if and only if $k \in A$. Then $E = \{ x_A \in \ell^\infty : A \in \mathcal{P}(\mathbf{N}) \}$ is uncountable and $A \neq B$ implies $\|x_A - x_B\|_\infty = 1$. Thus the balls $B_{\frac{1}{2}}(x_A)$ form an uncountable set of disjoint open balls. If D is dense, then D must meet every such ball and be uncountable. □

Subsequences

- Given a sequence (x_n), a subsequence is determined by choosing $\{ n_k \} \subset \mathbf{N}$ such that $n_k < n_{k+1}$ for all k. Then our subsequence is $(x_{n_k})_{k=1}^{\infty}$.
- To get a subsubsequence, we need $\{ k_j \} \subset \mathbf{N}$ such that $k_j < k_{j+1}$. Then we get $(x_{n_{k_j}})_{j=1}^{\infty}$. Clearly this is ugly and hard to grock even in LaTeX.
- Sometimes it is profitable to realize that a subsequence is determined by an infinite subset $S_1 = \{ n_1 < n_2 < n_3 < \cdots \} \subset \mathbf{N}$.
- Then a subsubsequence is determined by choosing an infinite subset $S_2 \subset S_1$: then $S_2 = \{ n_{k_1} < n_{k_2} < \cdots \}$. This makes it clear that a subsubsequence is actually a subsequence.

Notation

Remark

If $S_1 = \{ n_1 < n_2 < \cdots \}$ determines a subsequence (x_{n_k}), then we can write

$$\lim_{n \in S_1} x_n = a \quad \text{or} \quad (x_n)_{n \in S_1} \to a$$

is place of the old standby $\lim_{k \to \infty} x_{n_k} = a$. You should convince yourself that $\lim_{n \in S_1} x_n = a$ if and only if for all $\epsilon > 0$ there is a N such that $n \geq N$ and $n \in S_1$ implies that $\rho(x_n, a) < \epsilon$.

Lemma

Let (x_n) be a sequence in a metric space X and let S_1 be an infinite subset of $\mathbf{N}$ as above. Suppose that S_2 is an infinite subset of $\mathbf{N}$ such that $\{ n \in S_2 : n \notin S_1 \}$ is finite. Then if $\lim_{n \in S_1} x_n = a$, we also have $\lim_{n \in S_2} x_n = a$.

The Lemma

Theorem (Arzelà–Ascoli Lemma)

Suppose that X is a separable metric space. Let (f_n) be a pointwise bounded equicontinuous sequence in $C(X)$. Then (f_n) has a subsequence (f_{n_k}) such that $\lim_{k\to\infty} f_{n_k}(x)$ exists for all $x \in X$.

Proof.

Let $D = \{ x_i \}_{i=1}^{\infty}$ be (countable) dense subset of X. By assumption, $\big(f_n(x_1)\big)_{n\in\mathbf{N}}$ is a bounded sequence of complex numbers. Hence it has a convergent sequence determined by an infinite subset $S_1 \subset \mathbf{N}$. Let $a_1 = \lim_{n\in S_1} f_n(x_1)$. But $\big(f_n(x_2)\big)_{n\in S_1}$ is also bounded. Hence there is an infinite subset $S_2 \subset S_1$ such that $\lim_{n\in S_2} f_n(x_2) = a_2$. Furthermore, $\lim_{a\in S_2} f_n(x_1) = a_1$!

Proof

Proof Continued.

Continuing inductively we get $S_{k+1} \subset S_k$ such that

$$\lim_{m \in S_{k+1}} f_n(x_j) = a_j \quad \text{for all } 1 \leq j \leq k+1$$

Let r_k be the k^{th}-term in S_k. Let $S = \{ r_k \}_{k=1}^{\infty}$. Note that there are at most $k-1$ terms in S not in S_k. Thus by our sequence lemma,

$$\lim_{n \in S} f_n(x_j) = a_j \quad \text{for all } j.$$

At this point, we can replace $(f_n)_{n \in \mathbf{N}}$ by $(f_n)_{n \in S}$ and assume from here on that

$$\lim_{n \in \mathbf{N}} f_n(x_j) = a_j \quad \text{for all } x_j \in D.$$

Proof

Proof Continued.

Let $x_0 \in X$. Since x_0 is arbitrary, it will suffice to prove that $\big(f_n(x_0)\big)$ is Cauchy. Fix $\epsilon > 0$. Since (f_n) is equicontinuous at x_0, there is a $\delta > 0$ such that $\rho(x, x_0) < \delta$ implies $|f_n(x) - f_n(x_0)| < \epsilon/3$ for all $n \geq 1$. Since D is dense in X, there is a j such that $\rho(x_j, x_0) < \delta$. Since $\big(f_n(x_j)\big)$ is convergent, it is Cauchy. Hence there is a N such that $n, m \geq N$ implies that $|f_n(x_j) - f_m(x_j)| < \epsilon/3$. Now if $n, m \geq N$, we have

$$\begin{aligned} |f_n(x_0) - f_m(x_0)| \leq |f_n(x_0) - f_n(x_j)| + |f_n(x_j) - f_m(x_j)| + \\ |f_m(x_j) - f_m(x_0)| \\ < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

The Theorem

Remark

Suppose that X is a compact metric space. Then $C(X) = C_b(X)$ is a complete metric space with respect to the uniform norm. Moreover, by a homework problem, X is separable and we can apply the Arzelà–Ascoli lemma to $C(X)$.

Theorem (The Arzelà–Ascoli Theorem)

Let X be a compact metric space. Let (f_n) be a pointwise bounded equicontinuous sequence. Then (f_n) has a subsequence converging uniformly to some $f \in C(X)$.

Remark

It is interesting to note that a uniformly convergent sequence in $C(X)$ is necessarily uniformly bounded (by a homework problem). Hence our pointwise bounded equicontinuous sequence above must actually be uniformly bounded.

Preliminary Result

Lemma

Suppose that (X, ρ) is compact and that $\mathcal{F} \subset C(X)$ is equicontinuous on X. Then $\mathcal{F}$ is uniformly equicontinuous in that for all $\epsilon > 0$ there is a δ such that $\rho(x, y) < \delta$ implies that $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$.

Proof.

We will leave this as a homework exercise—see the proof that continuous functions on compact spaces are necessarily uniformly continuous. □

Proof of the AA Theorem.

Since compact metric spaces are separable, the AA Lemma applies and we can assume (f_n) has a subsequence (f_{n_k}) such that $f_{n_k}(x) \to f(x)$ for all $x \in X$ and some function f on X. To ease the notational burden, there is no harm in replacing (f_n) with this subsequence so that $f_n(x) \to f(x)$ for all $x \in X$. In particular, we can assume $\big(f_n(x)\big)$ is Cauchy for all $x \in X$. Since $C(X)$ is complete, we just have to show that (f_n) is uniformly Cauchy. That is, given $\epsilon > 0$, we want to find N such that $n, m \geq N$ implies

$$|f_n(x) - f_m(x)| < \epsilon \quad \text{for all } x \in X.$$

Proof

Proof Continued.

By uniform equicontinuity, there is a $\delta > 0$ such that $\rho(x, y) < \delta$ implies that $|f_n(x) - f_n(y)| < \epsilon/3$ for all $n \geq 1$. Since X is totally bounded, there are $x_1, \dots, x_n \in X$ such that $\{ B_\delta(x_j) \}_{j=1}^n$ covers X. Then there is a N such that $n, m \geq N$ implies $|f_n(x_j) - f_m(x_j)| < \epsilon/3$ for all $1 \leq j \leq n$. Now if $x \in X$ and $n, m \geq N$, there is a j such that $x \in B_\delta(x_j)$. Then

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f_n(x_j)| + |f_n(x_j) - f_m(x_j)| \\ &\qquad + |f_m(x_j) - f_m(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Corollary

Let X be a compact metric space. Suppose that $\mathcal{F} \subset C(X)$ be closed in $C(X)$ as well as equicontinuous on X and pointwise bounded. Then $\mathcal{F}$ is a compact subset of $C(X)$. In particular, $\mathcal{F}$ is uniformly bounded.

Proof.

It suffices to see that $\mathcal{F}$ is sequentially compact. Let (f_n) be a sequence in $\mathcal{F}$. By the AA-Theorem, (f_n) has a uniformly convergent subsequence (f_{n_k}). Since $\mathcal{F}$ is closed, $f = \lim_k f_{n_k} \in \mathcal{F}$. Hence $\mathcal{F}$ is compact.

Since $\mathcal{F}$ is compact in $C(X)$, it must be bounded (with respect to the metric induced from $\| \cdot \|_\infty$). Hence $\mathcal{F}$ is uniformly bounded. □

Compactness Theorem

Theorem

Suppose that (X, ρ) is a compact metric space. Then $\mathcal{F} \subset C(X)$ is compact if and only if $\mathcal{F}$ is closed, uniformly bounded, and equicontinuous.

Proof.

($\Longleftarrow$): This direction follows from the Corollary. In fact, we can replace uniformly bounded with pointwise bounded.

The Converse

Proof.

($\Longrightarrow$): Now we assume that $\mathcal{F}$ is compact. Then $\mathcal{F}$ is closed in $C(X)$ (this is a homework exercise). Furthermore, $\mathcal{F}$ must be bounded in $C(X)$ and therefore uniformly bounded. The real issue here is to see that $\mathcal{F}$ is equicontinuous. Assume to the contrary that $\mathcal{F}$ is not equicontinuous at $x \in X$. That means the statement

$$\forall \epsilon > 0, \exists \delta > 0, \text{s.t. } \rho(x, y) < \delta \Longrightarrow |f(x) - f(y)| < \epsilon \ \forall f \in \mathcal{F}$$

is false. Hence there is a $\epsilon_0 > 0$ such that for all $\delta > 0$, the above implication is false. Hence for all $n \geq 1$ there is a $x_n \in X$ and $f_n \in \mathcal{F}$ such that $\rho(x_n, x) < \frac{1}{n}$ and

$$|f_n(x_n) - f_n(x)| \geq \epsilon_0.$$

Proof

Proof Continued.

But $\mathcal{F}$ is compact, so (f_n) has a convergent subsequence (f_{n_k}) converging uniformly to $f \in C(X)$. Since f is continuous, $f(x_{n_k}) \to f(x)$. Furthermore there is a N such that $k \geq N$ implies $\|f_{n_k} - f\|_\infty < \epsilon_0/3$. Then

$$\begin{aligned} |f(x_{n_k}) - f(x)| &= |f(x_{n_k}) - f_{n_k}(x_{n_k}) + f_{n_k}(x_{n_k}) - f_{n_k}(x) \\ &\qquad + f_{n_k}(x) - f(x)| \\ &\geq |f_{n_k}(x) - f_{n_k}(x_{n_k})| \\ &\qquad - |f(x_{n_k}) - f_{n_k}(x_{n_k}) + f_{n_k}(x) - f(x)| \\ &\geq \epsilon_0 - \left(\frac{\epsilon_0}{3} + \frac{\epsilon_0}{3}\right) = \frac{\epsilon_0}{3} > 0 \end{aligned}$$

But this eventually contradicts $f(x_{n_k}) \to f(x)$. □

Break Time

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- Questions?
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Baire Spaces

Definition

A metric space is called a Baire space if the countable interection of open dense sets is dense. That is, if O_n is open and dense in X for all $n \geq 1$, then

$$\bigcap_{n=1}^{\infty} O_n$$

is dense in X.

Remark

This is a purely topological property. If ρ and σ are equivalent metrics on X, then (X, ρ) is Baire if and only if (X, σ) is Baire.

Definition

If E is a subspace of a metric space X, then the interior of E is

$$\operatorname{Int}(E) = \bigcup \{ U \subset X : U \subset E \text{ and } U \text{ is open in } X \}.$$

Remark

The interior $\operatorname{Int}(E)$ is the largest open set in X contained in E.

Example

Viewed as a subset of $\mathbf{R}$, $\operatorname{Int}(\mathbf{Q}) = \emptyset$. Let $E = [0, 1)$. Then in $\mathbf{R}$, $\operatorname{Int}(E) = (0, 1)$. But viewed as a subset of $X = [0, 1]$, then $\operatorname{Int}(E) = E$.

An Equivalent Formulation

Lemma

A metric space X is Barie if and only if given countably many closed sets $\{ F_n \}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} F_n$ has nonempty interior in X, then at least one of the F_n has nonempty interior.

Proof of the Lemma

Proof.

Note that a closed set F_n has nonempty interior if and only if $O_n = X \setminus F_n$ is open and dense. Furthermore, $X \setminus \bigcup F_n = \bigcap O_n$. Thus if $\bigcup F_n$ has interior, $\bigcap O_n$ is not dense.

Suppose that X is Baire. Let $\{ F_n \}$ be such that $\bigcup F_n$ has interior. If each F_n has empty interior, then each O_n is dense and $\bigcap O_n$ would be dense. This contradicts our assumption on $\bigcup F_n$. Hence some F_n has interior as required.

On the other hand, suppose that X has the property described in the statement of the lemma. Suppose that X is not Baire. Then there are dense open sets O_n such that $\bigcap O_n$ is not dense. But then $\bigcup F_n$ has interior. Hence one of the F_n has interior and the corresponding O_n would not be dense. This contradicts our assumptions and completes the proof. □

A Curiosity

Definition

A point in a metric space X is called isolated is $\{x\}$ is open in X.

Example

The subspace $\mathbf{Z} \subset \mathbf{R}$ consists entirely of isolated points. The interval $[0,1]$ has no isolated points.

Lemma

Let X be a Baire space without isolated points. Then X is uncountable.

Proof.

Suppose that $X = \{x_n\}_{n=1}^{\infty}$. Let $F_n = \{x_n\}$ so that each F_n is closed $X = \bigcup F_n$. Since X is open, the union has interior and hence some F_n has interior. But then $\{x_n\}$ is open and x_n is isolated. □

Next Time

Theorem (Baire Category Theorem)

Every complete metric space is a Baire space.

Remark

This applies to any metric space that admits an equivalent complete metric or to any metric space homeomorphic to a complete metric space. For example, $(0, 1)$ is homeomorphic to $\mathbf{R}$. Hence it is a Baire space in its standard topology.

That's Enough for Today

- That is enough for now.