Math 73/103: Fall 2020
Lecture 6

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Getting Started

- We should be recording!
- This a good time to ask questions about the previous lecture, complain, or tell a story.
- I hope that you have the bandwidth to keep your video on during the class meeting.
- Since Monday is part of Yom Kippur, we will cancel lecture on Monday (September 28th).

Last Time

- Recall that a metric space is Baire if the countable intersection of open dense sets is dense.
- Equivalently, a metric space is Baire if whenever the countable union of closed sets has interior, then at least one of the closed sets has interior.
- Our first goal today is to prove . . .

Theorem (Baire Category Theorem)

Every complete metric space is a Baire space.

Preliminary Result

Lemma

Suppose that U is open in a metric space (X, ρ) and that $x_0 \in U$. Then there is a $\delta > 0$ such that $\overline{B_\delta(x_0)} \subset U$.

Proof.

Since U is open, there is a $\delta > 0$ such that $B_{2\delta}(x_0) \subset U$. If $x \in \overline{B_\delta(x_0)}$, then there is a sequence $(x_n) \subset B_\delta(x_0)$ such that $x_n \to x$. But then $\rho(x_n, x_0) \to \rho(x, x_0)$. (Why?) This means $\rho(x, x_0) \leq \delta$. Therefore $\overline{B_\delta(x_0)} \subset B_{2\delta}(x_0) \subset U$ as required. □

Proof

Proof of the Baire Category Theorem.

Suppose that O_n is open and dense in X for all $n \geq 1$. Fix $x_0 \in X$ and $r_0 > 0$. It will suffice to show that

$$B_{r_0}(x_0) \cap \bigcap_{n=1}^{\infty} O_n \neq \emptyset.$$

Since O_1 is dense, $B_{r_0}(x_0) \cap O_1 \neq \emptyset$. Hence using our lemma, there is a $0 < r_1 < 1$ and $x_1 \in X$ such that $\overline{B_{r_1}(x_1)} \subset B_{r_0}(x_0) \cap O_1$.

But $B_{r_1}(x_1) \cap O_2 \neq \emptyset$ and there is $0 < r_2 < \frac{1}{2}$ and x_2 such that $\overline{B_{r_2}(x_2)} \subset B_{r_1}(x_1) \cap O_2$.

Proof

Proof Continued.

Continuing inductively, we get a sequence (x_n) in X and (r_n) in $(0,\infty)$ such that $0 < r_n < \frac{1}{n}$ and

$$\overline{B_{r_{n+1}}(x_{n+1})} \subset B_{r_n}(x_n) \cap O_{n+1}.$$

Let $F_n = \overline{B_{r_n}(x_n)}$. Then $F_{n+1} \subset F_n \cap O_{n+1} \subset F_n$ and $\operatorname{diam}(F_n) \searrow 0$. Since X is complete, it has the nested set property, and there is a $y_0 \in X$ such that $\{ y_0 \} = \bigcap F_n$.

Note that $y_0 \in F_1 \subset B_{r_0}(x_0)$ and $y_0 \in \bigcap O_n$.

This completes the proof. □

Examples

- Since $[0,1]$ is complete and has no isolated points, it must be uncountable.
- Let $X = \mathbf{R}^2$ and L_m the line $y = mx$. Then $O_m = \mathbf{R}^2 \setminus L_m$ is open and dense in the complete metric space $\mathbf{R}^2$. Hence $C = \bigcap_{r\in\mathbf{Q}} O_r = \mathbf{R}^2 \setminus \bigcup L_r$ is dense in $\mathbf{R}^2$. Can you show that C is necessarily uncountable?

The Boundary of a Set

Definition

If X is a metric space and $E \subset X$, then we say that $x \in X$ is a boundary point of E if $B_r(x)$ meets both E and $X \setminus E$ for all $r > 0$. The set of boundary points of E is denoted by ∂E.

Example

If $E = [0, 1) \subset \mathbf{R}$, then $\partial E = \{ 0, 1 \}$. On the other hand, if $E = \mathbf{Q} \subset \mathbf{R}$, then $\partial \mathbf{Q} = \mathbf{R}$.

Lemma

If X is a metric space and $E \subset X$, then ∂E is closed. If in addition, E is closed, then ∂E has empty interior.

Proof.

We will leave this as homework. □

Boundaries Again

Proposition

Let $\{ F_n \}_{n=1}^{\infty}$ be a countable collection of closed sets in a complete metric space X. Then $\bigcup_{n=1}^{\infty} \partial F_n$ has empty interior. In particular, $X \setminus \bigcup \partial F_n$ is dense.

Proof.

Since ∂F_n has empty interior for all n, this result follows immediately from the Baire Category Theorem and our characterization of Baire spaces in terms of closed sets. Since $X \setminus \bigcup \partial F_n = \bigcap X \setminus \partial F_n$, the latter is dense since each $X \setminus \partial F_n$ is open and dense. □

License Agreement

Remark

Your analyst license requires that you know that that uniform limit of continuous functions is necessarily continuous. You also should have a ready example to show that this can fail if we only have pointwise convergence. But we can use the above result to see that the pointwise limit of continuous real-valued functions is nevertheless continuous on a dense subset.

A Fun Theorem

Theorem (Royden & Fitzpatrick §10.2.7)

Suppose that X is a complete metric space. Let $(f_n) \subset C(X)$ converge pointwise to $f : X \to \mathbf{C}$. Then there is a dense set $D \subset X$ such that f is continuous at each point $x \in D$.

Remark

Royden & Fitzpatrick's proof will first show a stronger statement that $\mathcal{F} = \{ f_n \}$ is equicontinuous on D. This will imply that f is continuous on D. But to me, the continuity on D is the interesting bit.

The Proof

Proof.

For $n, m \in \mathbf{N}$, let

$$E(m,n) = \bigcap_{j,k \geq n} \{\, x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \,\}$$

$$= \{\, x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \text{ for all } j,k \geq n \,\}.$$

Note that $E(m,n)$ is closed. As we just observed, it follows that

$$D := X \setminus \Big(\bigcup_{n,m \in \mathbf{N}} \partial E(n,m) \Big)$$

is dense in X. Furthermore, if $x \in D \cap E(m,n)$, then x is in the the interior of $E(m,n)$.

Proof Continued.

We want to show that (f_n) is equicontinuous on D. Let $x_0 \in D$ and fix $\epsilon > 0$. Let $m \in \mathbf{N}$ be such that $0 < \frac{1}{m} < \frac{\epsilon}{4}$. Since $(f_n(x_0))$ converges, it is Cauchy. Let $N \in \mathbf{N}$ be such that $j, k \geq N$ implies $|f_j(x_0) - f_k(x_0)| \leq \frac{1}{m}$. Then $x_0 \in E(m, N)$. As observed on the previous slide, we must have x_0 in the interior of $E(m, N)$. Hence there is a $r > 0$ such that $B_r(x_0) \subset E(m, N)$. Thus

$$|f_j(x) - f_k(x)| \leq \frac{1}{m} \quad \text{for all } j, k \geq N \text{ and } x \in B_r(x_0).$$

Since f_N is continuous, we can find $0 < \delta < r$ so that

$$|f_N(x) - f_N(x_0)| < \frac{1}{m} \quad \text{if } x \in B_\delta(x_0).$$

return

Proof Continued.

Now if $j \geq N$ and $x \in B_\delta(x_0)$, we have

$$|f_j(x) - f_j(x_0)| \leq |f_j(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)|$$

which, in view of the ▸ last slide, is

$$\leq \frac{1}{m} + \frac{1}{m} + \frac{1}{m} = \frac{3}{m} < \frac{3}{4}\epsilon.$$

Since each f_j with $1 \leq j < N$ is continuous, we can shrink δ is necessary so that $|f_j(x) - f_j(x_0)| < \frac{3}{4}\epsilon$ for all j provided $x \in B_\delta(x_0)$. Hence (f_n) is equicontinuous at x_0. But now if $x \in B_\delta(x_0)$, we have

$$|f(x) - f(x_0)| = \lim_j |f_j(x) - f_j(x_0)| \leq \frac{3}{4}\epsilon < \epsilon.$$

Therefore f is continuous at x_0. Since $x_0 \in D$ was arbitrary, we are done. □

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

Back to Calculus

Let's consider the complete metric space $C\big([0,1],\mathbf{R}\big)$. We've been working over $\mathbf{C}$, but there is no harm in restricting to real-valued functions. If you insist, $C\big([0,1],\mathbf{R}\big)$ is a closed metric subspace of $C([0,1])$. Then if $f\in C\big([0,1],\mathbf{R}\big)$, we can define

$$D^+f(x)=\lim_{h\searrow 0}\frac{f(x+h)-f(x)}{h}$$

provided the limit exists and $x\in[0,1)$. Similarly, let

$$D^-f(x)=\lim_{h\nearrow 0}\frac{f(x+h)-f(x)}{h}$$

provided the limit exists and $x\in(0,1]$. Note that f is differentiable at $x\in(0,1)$ if and only if both limits exist and $f'(x)=D^+f(x)=D^-f(x)$. We say that f is nowhere differentiable if $f'(x)$ fails to exist for all $x\in[0,1]$ (where $f'(0)=D^+f(0)$ and $f'(1)=D^-f(1)$).

Big Theorem

Theorem

There is a nowhere differentiable function in $C\big([0,1],\mathbf{R}\big)$*. In fact, the collection of nowhere differentiable functions in* $C\big([0,1],\mathbf{R}\big)$ *is dense in the uniform norm.*

Lemma 1

Lemma

Suppose that $f \in C([0,1],\mathbf{R})$ *and* $f'(x_0)$ *exists for some* $x_0 \in [0,1]$. *Then there is a* $n \in \mathbf{N}$ *such that*

$$|f(x) - f(x_0)| \le n|x - x_0| \quad \text{for all } x \in [0,1].$$

▸ *return*

Proof.

Since f is continuous and $[0,1]$ is compact, there is a $M > 0$ such that $|f(x)| \le M$ for all $x \in [0,1]$. But there is a $\delta > 0$ such that $x \in [0,1]$ and $0 < |x - x_0| \le \delta$ implies $\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \le |f'(x_0)| + 1$. But if $|x - x_0| \ge \delta$ and $x \in [0,1]$, then $\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \le \frac{2M}{\delta}$. Hence we can take $n \in \mathbf{N}$ such that $n \ge \max\{\, |f'(x_0)| + 1, \frac{2M}{\delta}\,\}$. □

The $\mathcal{F}_n$'s

Definition

For each $n \in \mathbf{N}$ let $\mathcal{F}_n$ be the set of $f \in C\big([0,1],\mathbf{R}\big)$ such that there is a $x_f \in [0,1]$ such that $|f(x) - f(x_f)| \leq n|x - x_f|$ for all $x \in [0,1]$.

Lemma

Each $\mathcal{F}_n$ is closed in $C\big([0,1],\mathbf{R}\big)$. Moreover, if $f \in C\big([0,1],\mathbf{R}\big)$ is differentiable at at least one point, then $f \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$

The observation will be useful in proving the lemma and is of some interest on its own.

Lemma

Suppose that $(f_n) \subset C\big([0,1], \mathbf{R}\big)$ *converges uniformly to* $f \in C\big([0,1], \mathbf{R}\big)$. *Then if* $x_n \to x$ *in* X *we also have* $f_n(x_n) \to f(x)$.

Proof.

$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|.$ □

Proof of the $\mathcal{F}_n$ Lemma

Proof of the Lemma.

Suppose that $(f_n) \subset \mathcal{F}_n$ and that $f_n \to f$ in $C\big([0,1], \mathbf{R}\big)$. Let $x_n = x_{f_n}$. Since $[0,1]$ is compact, we can find a subsequence $x_{n_k} \to x_0$ in $[0,1]$. Since $f_{n_k} \to f$ uniformly, the preliminary lemma implies that $f_{n_k}(x_{n_k}) \to f(x_0)$. But then for all $y \in [0,1]$ we have

$$\begin{aligned} |f(y) - f(x_0)| &= \lim_k |f_{n_k}(y) - f_{n_k}(x_{n_k})| \leq \lim_k n|y - x_{n_k}| \\ &= n|y - x_0|. \end{aligned}$$

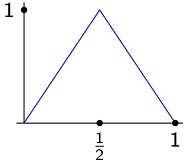
Thus $\mathcal{F}_n$ is closed. The second assertion follows from a previous lemma. $\square$

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

A function in $C([0,1],\mathbf{R})$ is called piecewise linear if there is a partition $P = \{ 0 = x_0 < x_1 < \cdots < x_n = 1 \}$ of $[0,1]$ such that f is linear on each $[x_{k-1}, x_k]$. For example, consider

$$\varphi(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \text{ and} \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$



Note that if f is piecewise linear then $D^{\pm} f(x)$ exists for all $x \in [0,1]$ (with appropriate caveats at 0 and 1). For example, $|D^{\pm}\varphi(x)| = 2$ for all x. The collection of piecewise linear functions is a vector subspace of $C([0,1],\mathbf{R})$. We let PW_n be the collection of piecewise linear functions in $C([0,1],\mathbf{R})$ for which $|D^{\pm}(f)(x)| \ge n$ for all $x \in [0,1]$.

Example

Let $\varphi_n(x) = \frac{1}{2^n}\varphi(4^n x)$. Then $\varphi_n \in \mathrm{PW}_{2^n}$ and $\|\varphi_n\|_\infty \le \frac{1}{2^n}$.

Lemma 2

Lemma

Suppose that $f \in C\big([0,1], \mathbf{R}\big)$ *and* $\epsilon > 0$. *Then for all* $n \in \mathbf{N}$, *there is a* $g \in \mathrm{PW}_n$ *such that* $\|f - g\|_\infty < \epsilon$.

Proof.

Since f is uniformly continuous, there is a $m \in \mathbf{N}$ such that $|x - y| < \frac{1}{m}$ implies $|f(x) - f(y)| < \epsilon/2$. Now let $x_i = \frac{i}{m}$ for $0 \leq i \leq m$. Define $g_0 : [0,1] \to \mathbf{R}$ by

$$g_0(\lambda x_i + (1-\lambda)x_{i+1}) = \lambda f(x_i) + (1-\lambda) f(x_{i+1})$$

for $\lambda \in [0,1]$ and $0 \leq i \leq m-1$. Then g_0 is continuous and piecewise linear on $[0,1]$ and is such that $\|f - g_0\|_\infty < \epsilon/2$.

Proof

Proof Continued.

Since both D^+g_0 and D^-g_0 take only finitely many values, there is a M such that $|D^{\pm}g_0(x)| \leq M$ for all $x \in [0,1]$. Then there is a $k \in \mathbf{N}$ such that $2^k \geq M + n$ and $2^{-k} < \frac{\epsilon}{2}$. Let $g = g_0 + \varphi_k$. Then $g \in \mathrm{PW}_n$ and $\|f - g\|_\infty < \epsilon$. □

Proof of the Theorem

Proof of the Theorem.

We have established that each $\mathcal{F}_n$ is closed. The lemma we just proved implies that no $\mathcal{F}_n$ has interior. Therefore $O_n = C\big([0,1],\mathbf{R}\big) \setminus \mathcal{F}_n$ is open and dense. Then

$$C\big([0,1],\mathbf{R}\big) \setminus \Big(\bigcup_{n=1}^{\infty} \mathcal{F}_n\Big) = \bigcap_{n=1}^{\infty} O_n$$

is dense and consists of nowhere differentiable functions in $C\big([0,1],\mathbf{R}\big)$. □

That's Enough for Today

- That is enough for now.