Math 73/103: Fall 2020
Lecture 8

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Getting Started

- We should be recording!
- This a good time to ask questions about the previous lecture, complain, or tell a story.
- There was no lecture 7—we had Monday off—so I am calling this lecture 8.
- I will not be able to hold office hours tomorrow (Thursday) from 10:30 to 11:30. Instead, we can use our x-hour 1:40 to 2:30.

Fixed Points

Definition

A function $f : X \to X$ has a fixed point if there is a $x_0 \in X$ such that $f(x_0) = x_0$.

Example

If V is a real or complex vector space, then a linear map $T : V \to V$ always has $\mathbf{0}_V$ as a fixed point. Such a T has a non-zero fixed point if and only if $\lambda = 1$ is an eigenvalue.

Examples of Fixed Point Results

Example

1. The function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x + 1$ has no fixed points.
2. However, if $f : [0,1] \to [0,1]$ is continuous, then I claim f always has a fixed point.
3. Proof: Let $g(x) = f(x) - x$. Then g is continuous on $[0,1]$. Furthermore $g(0) \geq 0$ while $g(1) \leq 0$. By the Intermediate Value Theorem, there is a $x_0 \in [0,1]$ such that $g(x_0) = 0$. The assertion follows.
4. The Brouwer Fixed Point Theorem gives the same result for functions $f : [0,1]^n \to [0,1]^n$ (and much more), but the techniques are not immediately accessible here.

Lipschitz Mappings

Definition

A function $f : (X, \rho) \to (X, \rho)$ is called Lipschtiz with Lipschitz constant M if there is a M such that

$$\rho\big(f(x), f(y)\big) \leq M\rho(x, y) \quad \text{for all } x, y \in X. \tag{1}$$

If we can take $M < 1$ in (1), then we call f a contraction.

Example

Suppose that $f : [a, b] \to [a, b]$ is continuous, and $f'(x)$ exists with $|f'(x)| \leq M$ for all $x \in (a, b)$. Then by the Mean Value Theorem, $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$. Hence f is Lipschitz with constant M (for the usual metric on $[a, b]$).

Example

Note that we can have $\rho\big(f(x), f(y)\big) < \rho(x, y)$ for all $x, y \in X$ without f being a contraction.

Banach Contraction Principle

Theorem (Banach Contraction Principle)

Suppose (X, ρ) is a complete metric space and that $f : (X, \rho) \to (X, \rho)$ is a contraction. Then f has a unique fixed point.

Proof.

Suppose that $M < 1$ is such that $\rho\big(f(x), f(y)\big) \le M\rho(x, y)$ for x, y. Fix $x_0 \in X$. Let (x_n) be the sequence such that $x_1 = f(x_0)$ and for $n \ge 1$ let $x_{n+1} := f(x_n)$.

I claim that it will suffice to see that (x_n) converges. To verify the claim, suppose that $x_n \to x$. Then $f(x) = \lim_n f(x_n) = \lim_n x_{n+1} = x$. Therefore x is a fixed point for f. If y is another fixed point, then $\rho(x, y) = \rho\big(f(x), f(y)\big) \le M\rho(x, y)$. Since $M < 1$, we must have $\rho(x, y) = 0$ and hence $x = y$. Thus is will suffice to prove that (x_n) converges.

Proof

Proof of the Claim.

Since X is complete, it will suffice to see that (x_n) is Cauchy. Note that $\rho(x_{n+1}, x_n) = \rho\big(f(x_n), f(x_{n-1})\big) \leq M\rho(x_n, x_{n-1})$ for all $n \geq 1$. By an induction argument, $\rho(x_{n+1}, x_n) \leq M^n \rho(x_1, x_0)$. Thus if $m > n$, we have

$$\begin{aligned}
\rho(x_m, x_n) &\leq \rho(x_m, x_{m-1}) + \rho(x_{m-1}, x_{m-2}) + \cdots + \rho(x_{n+1}, x_n) \\
&\leq \big(M^{m-1} + M^{m-2} + \cdots + M^n\big)\rho(x_1, x_0) \\
&= M^n\big(M^{m-n-1} + M^{m-n-2} + \cdots + 1\big)\rho(x_1, x_0) \\
&= M^n \frac{1 - M^{m-n}}{1 - M}\rho(x_1, x_0) \\
&\leq M^n \frac{1}{M - 1}\rho(x_1, x_0) = \frac{M^n}{1 - M}\rho(x_1, x_0).
\end{aligned}$$

Since $M < 1$, $M^n \to 0$. It follows that (x_n) is Cauchy. □

Constructive

Remark

If you have an applied bent, then you should be very happy with this proof. It actually provides a recipe for finding the fixed point! You just pick your favorite $x_0 \in X$, and apply f repeatedly.

Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

An Pretty Application

Remark

An first-order ordinary differential equation with initial conditions is one of the form

$$\begin{cases} y' &= g(x, y) \\ y(x_0) &= y_0 \end{cases} \tag{2}$$

for some point $(x_0, y_0) \in \mathbf{R}^2$. Of course this is shorthand asking for a differentiable function y on an interval containing x_0 such that $y(x_0) = y_0$ and $y'(x) = g(x, y(x))$ near x_0. Of course, we want to tell our students that (2) has a unique solution. Let's see under what circumstances it does.

Sanity

Remark

A decent respect for our sanity suggests we want to restrict to the case that g is continuous in a neighborhood U of (x_0, y_0) in $\mathbf{R}^2$ (with the Euclidean metric coming from $\|\cdot\|_2$). Since $y(x) = 1/(1-x)$ is a solution to

$$\begin{cases} y' &= y^2 \\ y(0) &= 1, \end{cases}$$

and since y "blows up" at $x = 1$, we see that we can't hope to do better that find a solution near $x_0 = 0$. Furthermore, both $y \equiv 0$ and

$$y(x) = \begin{cases} 0 & \text{if } x \leq 0, \text{ and} \\ x^2 & \text{if } x \geq 0 \end{cases}$$

are solutions to $\begin{cases} y' &= 2\sqrt{y} \\ y(0) &= 0. \end{cases}$

Fundamental Theorem of Calculus

When teaching an ODE course, we always start out with the simple case where $g(x, y) = h(x)$ for some continuous function h. Then we can solve our initial value problem (IVP) the old fashioned way:

$$y(x) = y_0 + \int_{x_0}^{x} h(t)\,dt. \tag{3}$$

The Fundamental Theorem of Calculus implies (3) provides a solution and the Mean Value Theorem gives uniqueness.

We can upgrade this to observe that we can solve our general IVP if we can find a function $y : I \to \mathbf{R}$ defined on an interval I containing x_0 such that $(x, y(x)) \in U$ for all $x \in I$ and

$$y(x) = y_0 + \int_{x_0}^{x} g(t, y(t))\,dt.$$

Break Time

- Definitely time for a break.
- Questions?
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Existence and Uniqueness

Theorem (The Picard Local Existence Theorem)

Let U be an open neighborhood of $(x_0, y_0) \subset \mathbf{R}^2$. Suppose that $g : U \subset \mathbf{R}^2 \to \mathbf{R}$ is continuous and such that there is a $M > 0$ such that

$$\big|g(x, y_1) - g(x, y_2)\big| \leq M|y_1 - y_2| \text{for all } (x, y_1) \text{ and } (x, y_2) \text{ in } U.$$

Then there is an open interval I such that $x_0 \in I$ and such that there is unique differentiable function $y : I \to \mathbf{R}$ such that $y(x_0) = y_0$ and $y'(x) = g(x, y(x))$ for all $x \in I$.

Proof

Proof.

For all $\delta > 0$, let $I_\delta = [x_0 - \delta, x_0 + \delta]$. Since U is open, there are positive numbers a and b such that the rectangle $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$ is contained in U. For each $\delta > 0$, let X_δ be the subspace of $C(I_\delta, \mathbf{R})$ of functions f such that $|f(x) - y_0| \leq b$ for all $x \in I_\delta$. This just means that the graph of f is in the rectangle $I_\delta \times [y_0 - b, y_0 + b] \subset R \subset U$. It is not hard to check that X_δ is closed in $C(I_\delta, \mathbf{R})$. Define $T : X_\delta \to C(I_\delta, \mathbf{R})$ by

$$T(f)(x) = y_0 + \int_{x_0}^{x} g(t, f(t))\, dt \quad \text{for } x \in I_\delta.$$

It will suffice to find a unique f such that $T(f) = f$. Since X_δ is a complete metric space, it will suffice to see that there is δ such that $T(X_\delta) \subset X_\delta$ and such that $T : X_\delta \to X_\delta$ is a contraction.

Proof Continued.

Since R is compact, there is a K such that $|g(x,y)| \leq K$ for all $(x,y) \in R$. Thus if $f \in X_\delta$ and $x \in I_\delta$,

$$|T(f)(x) - y_0| = \Big| \int_{x_0}^{x} g(t, f(t))\, dt \Big| \leq \delta K$$

Hence $T(X_\delta) \subset X_\delta$ provided $\delta K \leq b$.

Proof

Proof Continued.

On the other hand, our assumptions on g are such that if $f_1, f_2 \in X_\delta$ and $x \in I_\delta$, then

$$|g(x, f_1(x)) - g(x, f_2(x))| \leq M|f_1(x) - f_2(x)| \leq M\|f_1 - f_2\|_\infty.$$

Therefore

$$\begin{aligned} \big|T(f_1)(x) - T(f_2)(x)\big| &= \Big|\int_{x_0}^{x} \big[g(t, f_1(t)) - g(t, f_2(t))\big]\,dt\Big| \\ &\leq |x - x_0| M |f_1(t) - f_2(t)| \leq \delta M \|f_1 - f_2\|_\infty \end{aligned}$$

Therefore $\|T(f_1) - T(f_2)\|_\infty \leq \delta M \|f_1 - f_2\|_\infty$. Thus if we let $\delta = \min\{\, b/K, \frac{1}{2}M \,\}$, then $T(X_\delta) \subset X_\delta$ and T is a contraction. □

That's Enough for Today

- That is enough for now.