# Math 73/103: Fall 2020
# Lecture 9

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# Getting Started

- We should be recording!
- This a good time to ask questions about the previous lecture, complain, or tell a story.
- Speaking of complaining, let's have homework problems 11 to 23 due Wednesday via `gradescope`.

# Completion

## Definition

Let $(X, \rho)$ be a metric space. We say that $f : (X, \rho) \to (X', \rho')$ a completion of $(X, \rho)$ if $(X', \rho')$ is a complete metric space and $f$ is isometric with $f(X)$ is dense in $X'$.

## Remark

If $f : (X, \rho) \to (X', \rho')$ is a completion, then since $f$ is isometric—that is, $\rho'(f(x), f(y)) = \rho(x, y)$—we can identify $(X, \rho)$ with the subspace $f(X)$ of $(X', \rho')$.

# Examples

## Example

1. Let $(X,\rho) = ((0,1), |\cdot|)$. Then we can let $(X',\rho') = ([0,1], |\cdot|)$.
2. Let $(X,\rho) = (\mathbf{Q}, |\cdot|)$. Then we can let $(X',\rho') = (\mathbf{R}, |\cdot|)$.
3. Now let $X = C([0,1])$ equipped with the metric $\rho(f,g) = \int_0^1 |f(t) - g(t)|\, dt$. It is easy to see $(X,\rho)$ is not complete. It is not so easy to find a completion. In a few weeks, we would happily answer $L^1([0,1])$ with respect to Lebesgue measure and the metric coming from the $\|\cdot\|_1$-norm.

# Uniqueness

**Proposition**

*Suppose that* $f : (X, \rho) \to (X', \rho')$ *and* $g : (X, \rho) \to (X'', \rho'')$ *are completions of* $(X, \rho)$*. Then there is a unique surjective isometry* $h$ *such that*

$$\begin{array}{ccc} (X', \rho') & \xrightarrow{\quad h \quad} & (X'', \rho'') \\ & {}_{f}\nwarrow \quad \nearrow_{g} & \\ & (X, \rho) & \end{array}$$

*commutes.*

# A Lemma

## Lemma

*Let $(Y, \sigma)$ be a complete metric space. Suppose that $D$ is dense in $(X, \rho)$, that $f : D \to Y$ is uniformly continuous. Then there is a unique uniformly continuous function $g : (X, \rho) \to (Y, \sigma)$ extending $f$.*

## Proof.

Let $x \in X$ and suppose $(d_n) \subset D$ converges to $x$. Since $f$ is uniformly continuous $(f(d_n))$ is Cauchy in $Y$. Suppose $(d_n)$ and $(d'_n)$ are both sequences in $D$ convering to $x$. Then there are $y, y' \in Y$ such that $f(d_n) \to y$ and $f(d'_n) \to y'$. Let $d''_{2k} = d_k$ and $d''_{2k+1} = d'_k$. Then $(d''_n)$ also converges to $x$ and $f(d''_n) \to y''$. Then $y = y'' = y'$ and we can define $g : X \to Y$ by letting $g(x) = \lim_n f(d_n)$ for any sequence $(d_n)$ in $D$ converging to $x$. Clearly, $g$ extends $f$.

**Proof Continued.**

Fix $\epsilon > 0$. Since $f$ is uniformly continuous on $D$, there is a $\delta > 0$ such that $\rho(d, d') < \delta$ implies $\sigma(f(d), f(d')) < \epsilon/2$. Now suppose that $\rho(x, y) < \delta$. Let $(d_n)$ and $(d_n')$ be sequences in $D$ converging to $x$ and $y$, respectively. Since $\rho(d_n, d_n') \to \rho(x, y) < \delta$, we eventually have $\rho(d_n, d_n') < \delta$. Hence we eventually have $\sigma(f(d_n), f(d_n')) < \epsilon/2$. Hence $\sigma(g(x), g(y)) \leq \epsilon/2 < \epsilon$. □

# Proof of the Proposition

## Proof of the Proposition.

Let $f : (X, \rho) \to (X', \rho')$ and $g : (X, \rho) \to (X'', \rho'')$ be completions. Since $f$ is isometric, it is injective and we can define $h_0 : f(X) \to g(X)$ by $h_0(f(x)) = g(x)$. Then $h_0$ is isometric and hence uniformly continuous. Thus there is a unique extension $h : (X', \rho') \to (X'', \rho'')$ and it is not hard to see that $h$ is isometric. We still want to see that $h$ is surjective. Since $g(X) \subset h(X')$, $h(X')$ is dense. If $x'' \in X''$, then there is a sequence $h(x_n) \to x''$. Since $h$ is isometric, $(x_n)$ must be Cauchy. Then $x_n \to x'$ in $X'$. But then $h(x_n) \to h(x') = x''$ and $h$ is onto. □

# Building a Completion

## Definition

if $(X, \rho)$ is a metric space, let $CS(X, \rho)$ be the set of all Cauchy sequences in $X$.

## Remark

Since we will want to look at sequences is $CS(X, \rho)$, we will view elements of $CS(X, \rho)$ as functions $x$ on $\mathbf{N}$ such that $(x(n))_{n=1}^{\infty}$ is Cauchy in $(X, \rho)$.

**Lemma**

*If* $x, y \in CS(X, \rho)$*, then* $\lim_{n\to\infty} \rho(x(n), y(n))$ *exists (in* $[0, \infty)$*).*

**Proof.**

It suffices to see that $\big(\rho(x(n), y(n)\big)$ is Cauchy in $\mathbf{R}$. But

$$\begin{aligned}|\rho(x(n), y(n)) - \rho(x(m), y(m))| \\ &\le |\rho(x(n), y(n)) - \rho(x(m), y(n))| \\ &\qquad + |\rho(x(m), y(n)) - \rho(x(m), y(m))| \\ &\le \rho(x(n), x(m)) + \rho(y(n), y(m)).\end{aligned}$$

The rest is straightforward. □

# Puesdo metric

**Lemma**

*Let $(X, \rho)$ be a metric space. Then $d(x, y) = \lim \rho(x(n), y(n))$ is a puesdo metric on $CS(X, \rho)$.*

**Proof.**

Clearly, $d$ is symmetric and $d(x, x) = 0$. If $x, y, z \in CS(X, \rho)$, then for each $n \in \mathbf{N}$, we have
$\rho(x(n), z(n)) \leq \rho(x(n), y(n)) + \rho(y(n), z(n))$. Taking limits gives
$d(x, z) \leq d(x, y) + d(y, z)$. □

# Equivalence Relation

## Definition

If $x, y \in CS(X, \rho)$, then we say that $x \sim y$ if $d(x, y) = 0$.

## Lemma

*The relation $x \sim y$ is an equivalence relation on $CS(X, \rho)$. If $X' = CS(X, \rho)/\sim$ is the set of equivalence classes $[x]$ with $x \in CS(X, \rho)$, then $\rho'([x], [y]) = d(x, y)$ is a well-defined metric on $X'$.*

# The Proof

**Proof.**

Since we certainly have $x \sim x$ for all $x$, and $x \sim y$ implies $y \sim x$, we just have to check transitivity. Suppose $x \sim y$ and $y \sim z$. Then

$$\begin{aligned} 0 \leq d(x,z) &= \lim_n \rho(x(n), z(n)) \\ &\leq \lim_n \big[\rho(x(n), y(n)) + \rho(y(n), z(n))\big] \\ &= 0. \end{aligned}$$

Hence $d(x,z) = 0$ and $x \sim z$.

# Proof

## Proof Continued.

If $x \sim x'$, then $0 \leq d(x, y) \leq d(x, x') + d(x', y) = d(x', y)$. By symmetry, we also have $d(x', y) \leq d(x, y)$. That is, $d(x, y) = d(x', y)$. Thus if we also have $y \sim y'$, then $d(x, y) = d(x', y) = d(x', y')$. It follows that $\rho'([x], [y]) = d(x, y)$ is well defined on $X'$.

Checking that $\rho'$ is a metric is not hard. For example, if $\rho'([x], [y]) = 0$, then $d(x, y) = 0$ and $x \sim y$. That is, $[x] = [y]$. □

# Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

# Existence

**Theorem**

*If $a \in X$, let $h(a) \in CS(X,\rho)$ be the constant sequence $h(a)(k) = a$ for all $k$. Then $f : (X,\rho) \to (X',\rho')$ with $f(a) = [h(a)]$ is a completion of $(X,\rho)$.*

**Remark**

We'll break the proof up into a number of lemmas. We have to prove that $f$ is isometric with dense range, and that $(X',\rho')$ is complete.

# Isometric with Dense Range

## Lemma

*The map* $f : (X, \rho) \to (X', \rho')$ *with* $f(a) = [h(a)]$ *is isometric with dense range.*

## Proof.

We have $\rho'(f(a), f(b)) = \rho'([h(a)], [h(b)]) = d(h(a), h(b)) = \lim_k \rho(h(a)(k), h(b)(k)) = \rho(a, b)$. It follows that $f$ is isometric.

Now let $[y] \in X'$ and $\epsilon > 0$. Let $N$ be such that $n, m \geq N$ implies $\rho(y(n), y(m)) < \epsilon/2$. Then

$$\rho'(f(y(N)), [y]) = d(h(y(N)), y) = \lim_k \rho\big(y(N), y(k)\big) \leq \frac{\epsilon}{2} < \epsilon.$$

Thus $f(X)$ is dense in $X'$. □

# Completeness

## Proposition

*The metric space $(X', \rho')$ is complete.*

## Proof.

Suppose that $([x_n])$ is Cauchy in $(X', \rho')$. Since $f(X)$ is dense in $X'$, there is a $a_n \in X$, such that $\rho'([h(a_n), [x_n]) = d(h(a_n), x_n) < \frac{1}{n}$.

Now define a sequence $y$ in $X$ by $y(k) = a_k$. I claim that $y \in CS(X, \rho)$.

**Proof Continued.**

Let $\epsilon > 0$. Let $N$ be such that $n, m \geq N$ implies $\rho'([x_n], [x_m]) = d(x_n, x_m) < \frac{\epsilon}{3}$. We can also assume that $\frac{1}{N} < \frac{\epsilon}{3}$. Since $f(a) = [h(a)]$ is isometric, it follows that if $n, m \geq N$, then

$$\begin{aligned}\rho(y(n), y(m)) &= \rho(a_n, a_m) \\ &= \rho'([h(a_n)], [h(a_m]) = d(h(a_n), h(a_m)) \\ &\leq d(h(a_n), x_n) + d(x_n, x_m) + d(x_m, h(a_m)) \\ &< \frac{1}{n} + \frac{\epsilon}{3} + \frac{1}{m} < \epsilon.\end{aligned}$$

Hence $y \in CS(X, \rho)$ as claimed.

# Proof

**Proof Continued.**

Now it will suffice to see that $\lim_n \rho'([x_n],[y]) = \lim_n d(x_n, y) = 0$. Fix $\epsilon > 0$. Let $N$ be such that $n, m \geq N$ implies $\rho(y(n), y(m)) < \frac{\epsilon}{2}$ and $\frac{1}{N} < \frac{\epsilon}{2}$. Notice that if $n \geq N$ then $d(h(a_n), y) = \lim_k \rho(a_n, y(k)) = \lim_k \rho(a_n, a_k) \leq \frac{\epsilon}{2}$. Now if $n \geq N$,

$$d(x_n, y) \leq d(x_n, h(a_n)) + d(h(a_n), y)$$
$$\leq \frac{1}{n} + \frac{\epsilon}{2} < \epsilon.$$

This completes the proof that $(X', \rho')$ is complete. □

Combining the lemma and proposition gives a proof of the existence theorem. That is, $f : (X, \rho) \to (X', \rho')$ is a completion of $(X, \rho)$.

# Break Time

- Definitely time for a break.
- Questions?
- Start recording again.

# Time For Measure Theory

- Measure theory is the NC-17 version of integration.
- Our Calculus students would happily tell us that integration amounts to anti-differentiation.
- This is wrong. At its most basic level, integration is about areas under curves. However, we quickly learn that we can apply these techniques to applications such as computing various physical quantities, Fourier transforms, Laplace transforms, and . . . .
- More abstractly, and we will eventually get very abstract, an integral is a linear map on families of functions with good convergence properties.

# The Ghost of Riemann

- Of course, we come to Math 73/103 fully equipped with knowledge of the Riemann integral.
- Back in the day, my class of Berkeley graduate students would pass around sample questions from the dreaded analysis oral qualifying exam. One—of many—that troubled me was the following. "During your exam, the ghost of Riemann appears and inquires firmly but politely 'why all this fuss? What is wrong with my integral?' What do we tell Riemann?"

# A Hint

**Remark**

We are not yet ready to give a good answer to this question. But here is a hint. Consider a sequence $(f_n)$ of continuous functions $f_n : [0,1] \to [0,1]$ that converge pointwise to 0. Is it the case that

$$\int_0^1 f_n(x)\,dx \to 0?$$

Of course the answer is "no" if we don't assume that $f_n$ are bounded—let $f_n$ be the piecewise linear function that is 0 at 0, $n$ at $1/2n$, and identically 0 from $1/n$ to 1. But the answer is "yes" if we assume $f_n([0,1]) \subset [0,1]$ for all $n$! (In fact, all we need is for $(f_n)$ to be uniformly bounded.) This will be an easy consequence of results we will soon prove, but not if we restrict to Riemann techniques.

# What is the Riemann Integral?

- The Riemann integral is about bounded, real-valued functions $f$ on a closed and bounded interval $[a, b] \subset \mathbf{R}$.
- A finite set $P = \{ a = t_0 < t_1 < \cdots < t_n = b \}$ is called a partition of $[a, b]$.
- For $1 \leq k \leq n$, let $m_k = \inf_{t \in [t_{k-1}, t_k]} f(t)$, $M_k = \sup_{t \in [t_{k-1}, t_k]} f(t)$, and $\Delta t_k = t_k - t_{k-1}$.
- Then we define

$$\mathcal{U}(f, P) = \sum_{k=1}^{n} M_i \Delta t_k \quad \text{and} \quad \mathcal{L}(f, P) = \sum_{k=1}^{n} m_k \Delta t_k.$$

- As well as,

$$\overline{\mathcal{R}}\int_a^b f = \inf_P \mathcal{U}(f, P) \quad \text{and} \quad \underline{\mathcal{R}}\int_a^b f = \sup_Q \mathcal{L}(f, Q)$$

where $P$ and $Q$ vary over all partitions of $[a, b]$.

**Definition**

We say that a bounded real-valued function $f$ on $[a, b]$ is Riemann integrable if the upper Riemann integral $\overline{\mathcal{R}}\int_a^b f$ and the lower Riemann integral $\underline{\mathcal{R}}\int_a^b f$ are equal. In that case, we call the common value is denoted by $\mathcal{R}\int_a^b f$. The set of all Riemann integrable functions on $[a, b]$ is denoted by $\mathcal{R}[a, b]$.

# Characterization

**Proposition**

*A bounded real-valued function on $[a, b]$ is Riemann integrable if and only if for all $\epsilon > 0$ there is a partition $P$ of $[a, b]$ such that*

$$\mathcal{U}(f, P) - L(f, P) < \epsilon.$$

**Proof.**

I will leave this as a guided homework problem. □

# That's Enough for Today

- That is enough for now.