Math 73/103 Homework week 1

Last Updated: September 16, 2021

Wednesday 09/15/2021

1. Let E be a subset of a metric space (X, ρ). We say that x is a limit point of E if there is a sequence $(x_n) \subset E$ such that $x_n \to x$. Show that E is closed if and only if E contains all its limit points.

2. State and prove a result characterizing open sets in a metric space in terms of sequences (as we did for closed sets in the previous problem). The following terminology might be useful. If U is a subset of a metric space X, then a sequence $(x_n) \subset X$ is eventually in U if there is a N such that $n \geq N$ implies $x_n \in U$.

3. Let (X, ρ) be a metric space and fix $E \subset X$. For $x \in X$, we define

$$f(x) = \inf_{y \in E} \rho(x, y).$$

   1. Show that f is continuous.
   2. Show that $\{x : f(x) = 0\} = \overline{E}$.

4. Show that a metric space is separable if and only if the topology associated with its metric is second-countable.

5. A sequence (x_n) in a metric space (X, ρ) is a *bounded sequence* if its corresponding point set is bounded. Show that a convergent sequence in a metric space is bounded and its limit is unique.

6. Show that if (x_n) and (y_n) are convergent sequences in a metric space (X, ρ), then $\rho(x_n, y_n)$ converges.