Math 73/103 Homework week 3

Last Updated: September 28, 2021

Monday 09/27/2021

1. Show that a metric space (X, ρ) is compact if and only if every collection of closed sets with the Finite Intersection Property has non-empty intersection.

2. Suppose (X, ρ_X) and (Y, ρ_Y) are metric spaces.

   (a) Show that if $f : X \to Y$ is continuous and $K \subset X$ is compact, then $f(K)$ is compact in Y.

   (b) Show that if $f : X \to Y$ is a continuous bijection and X is compact, then f is a homeomorphism. Is f necessarily also an isometry?

3. Show that any continuous function $f : X \to Y$ from a compact metric X space into another metric space Y is uniformly continuous.

4. Prove Lebesgue's Covering Lemma: Let (X, ρ) be a compact metric space and let $U = \{U_i\}$ be an open cover of X. Show that there is a number $L(U) > 0$ (the Lebesgue number of U) such that any non-empty $S \subset X$ with diameter $\delta(S) < L(U)$ is contained in some U_i.