Math 73/103 Homework week 3

Last Updated: October 13, 2021

Monday 10/4/2021

1. Suppose that X is a complete metric space without isolated points, and that O_n is open and dense for all $n \geq 1$. Show that $\cap_{n\geq 1} O_n$ is uncountable. (Note that if x is not isolated, then $X \setminus \{x\}$ is open and dense.)

Wednesday 10/6/2021

2. Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of $[a, b]$. Prove that $L(f, \mathcal{P}) \leq U(f, \mathcal{Q})$, where $L(f, \mathcal{P})$ and $U(f, \mathcal{Q})$ are the lower and upper Riemann sums, respectively.

3. Prove that a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that
$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

4. Let X be an uncountable set and let $\mathcal{M}$ be the collection of all subsets E of X such that either E or E^c is countable. Show that $\mathcal{M}$ is a σ-algebra.