Math 73/103 Homework week 3

Last Updated: October 13, 2021

Monday 10/11/2021

1. Let (a_n) be a sequence in $[-\infty, \infty]$.

   (a) Show that $\liminf_n a_n \leq \limsup_n a_n$.

   (b) Suppose that $\lim a_n$ exists and equals $L \in [-\infty, \infty]$. Show that $\limsup_n a_n = L = \liminf_n a_n$.

   (c) Suppose that $\limsup_n a_n = L = \liminf_n a_n$. Show that $\lim_n a_n$ exists and equals L.

2. Suppose that $f, g : (X, \mathcal{M}) \to [-\infty, \infty]$ are measurable functions. Prove that the sets

$$\{ x : f(x) < g(x) \} \quad \text{and} \quad \{ x : f(x) = g(x) \}$$

   are measurable. (Remark: if $h = f - g$ were defined, then this problem would be much easier (why?). The problem is that $\infty - \infty$ and $-\infty + \infty$ make no sense, so h may not be everywhere defined.)

Wednesday 10/13/2021

3. Let X be an uncountable set and $\mathcal{M}$ the σ-algebra of subsets E of X such that either E or E^C is countable. Define $\mu : \mathcal{M} \to [0, \infty]$ by $\mu(E) = 0$ if E is countable and $\mu(E) = 1$ if E is uncountable.

   (a) Show that μ is a measure on $(X, \mathcal{M})$.

   (b) Describe the measurable functions $f : X \to \mathbf{C}$ and their integrals. (Hint: show that a measurable function must be constant off a countable set; that is, f must be constant μ-almost everywhere.)