Math 73/103 Homework week 7

Last Updated: October 31, 2021

Monday 10/25/2021

1. Suppse that $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$. Show that for all $\epsilon > 0$ there is a $\delta > 0$ such that $\mu(E) < \delta$ implies that
$$\int_E |f| \, d\mu < \epsilon.$$
(Hint: First work the problem assuming that f is bounded. Then let $E_n = \{ x : |f(x)| \leq n \}$ and $f_n = \chi_{E_n} \cdot f$. Then observe that $\int_X |f - f_n| \, d\mu \to 0$.)

2. Suppose that $\{A_n\}_{n=1}^{\infty}$ are subsets of a set X. Let $B_1 = A_1$ and $B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$ for $n \geq 2$. Show that the B_n are pairwise disjoint and that $\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k$ for all n. Observe that if $(X, \mathcal{M})$ is a measurable space and the A_n are measurable, then so are the B_n.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \mathcal{M}_0, \mu_0)$ be its completion.
    1. If $f : X \to \mathbf{C}$ is μ_0-measurable, show that there is a μ-measurable function $g : X \to \mathbf{C}$ such that $f = g$ for μ_0-almost all x. (Hint: show that it suffices to assume that f is a μ_0-measurable simple function.)
    2. Further observe that there is a μ-null set N such that $f(x) = g(x)$ if $x \notin N$.
    3. What does this result say about Lebesgue measurable functions on $\mathbf{R}$ and Borel functions?

Wednesday 10/27/2021

4. Suppose that Y is a topological space and that $\mathcal{M}$ is a σ-algebra in Y containing all the Borel sets. Suppose that ν is a measure on $(Y, \mathcal{M})$ such that for all $E \in \mathcal{M}$
$$\mu(E) = \inf\{ \mu(V) : V \text{ is open and } E \subset V \}. \tag{6}$$
Suppose also that
$$Y = \bigcup_{n=1}^{\infty} Y_n \quad \text{with } \mu(Y_n) < \infty \text{ for all } n \geq 1. \tag{$*$}$$
In this case we say that μ is a *σ-finite outer-regular measure* on $(Y, \mathcal{M})$.

1. Show that Lebesgue measure m on $(\mathbf{R}, \mathcal{L})$ is an example of a σ-finite outer-regular measure.
2. If $E \in \mathcal{M}$ and if $\epsilon > 0$, then show that there is an open set V and a closed set F such that $F \subset E \subset V$ with $\mu(V \setminus F) < \epsilon$. (Hint: first assume $\mu(E) < \infty$. Then use $(*)$.)
3. Recall that a countable intesection of open sets is called a G_δ-set, and that a countable union of closed sets is called a F_σ-set. Show that if $E \in \mathcal{M}$, then there is a G_δ-set G and a F_σ-set A such that $A \subset E \subset G$ and $\mu(G \setminus F) = 0$.
4. Use the above to conclude that $(\mathbf{R}, \mathcal{L}, m)$ is the completion of restriction of Lebesgue measure to the Borel subsets of $\mathbf{R}$.

5. We define the *symmetric difference* of two subsets E and F to be $E \Delta F := E \setminus F \cup F \setminus E$. In this problem, you may use without proof that every open subset of $\mathbf{R}$ is a countable disjoint union of open intervals. Suppose that $E \subset \mathbf{R}$ is a set of finite Lebesgue measure. Let $\epsilon > 0$. Show that there is finite disjoint union F of open intervals such that $m(E \Delta F) < \epsilon$.