Math 73/103 Homework week 8/9

Last Updated: November 12, 2021

1. Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and that $f : X \to [0, \infty]$ is measurable. Let ν be the measure on $(X, \mathcal{M})$ defined by

$$\nu(E) = \int_E f(x)\, d\mu(x) \quad \text{for all } E \in \mathcal{M}.$$

Show that if g is measurable then $g \in \mathcal{L}^1(X, \nu)$ if and only if $fg \in \mathcal{L}^1(X, \mu)$ and that

$$\int_X g(x)\, d\nu(x) = \int_X g(x) f(x)\, d\mu(x).$$

2. Suppose that ρ is a premeasure on an algebra $\mathcal{A}$ of sets in X.

   1. Show that

   $$\rho^*(E) = \inf\{\sum_{k=1}^{\infty} \rho(A_k) : \text{each } A_k \in \mathcal{A} \text{ and } E \subset \bigcup A_k\}$$

   is an outer measure on X.

   2. Show that $\rho^*(A) = \rho(A)$ for all $A \in \mathcal{A}$.
   3. Show that each $A \in \mathcal{A}$ is ρ^*-measurable.

3. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{M}, \nu)$ both be counting measure $(\mathbf{N}, \mathcal{P}(\mathbf{N}), \nu)$. Define $f : X \times Y \to \mathbb{R}$ by

$$f(m, n) = \begin{cases} 1 & \text{if } m = n, \\ -1 & \text{if } m = n + 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Show that f is not integrable and that the two iterated integrals $\int_Y \int_X f\, d\mu\, d\nu$ and $\int_X \int_Y f\, d\nu\, d\mu$ are not equal.

4. Let $\mathcal{B}(\mathbf{R})$ be the Borel σ-algebra in $\mathbf{R}$ and $\mathcal{B}(\mathbf{R}^2)$ the Borel σ-algebra in $\mathbf{R}^2$. Show that $\mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R}) = \mathcal{B}(\mathbf{R}^2)$. (You may use the observation that every open set in $\mathbf{R}^2$ is a countable union of open rectangles.)

5. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite *complete* measure spaces. Let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$.

   (a) Suppose that $E \in \mathcal{M} \otimes \mathcal{N}$ and $\mu \times \nu(E) = 0$. Show that $\mu(E^y) = 0 = \nu(E_x)$ for μ-almost all x and ν-almost all y.

(b) Suppose that f is $\mathcal{L}$-measurable and that $f(x, y) = 0$ for λ-almost all (x, y). Show that there is a μ-null set M and a ν-null set M such that for all $x \notin M$ and $y \notin N$, f_x and f^y are integrable and that

$$\int_X f^y \, d\mu = 0 = \int_Y f_x \, d\nu.$$