**Homework problems, due May 4, 2009**

1. Let $\varphi$ denote Euler's function. Show that for each positive integer $m$, we have that the set $\{n : m \mid \varphi(n)\}$ has asymptotic density 1.

2. Let $\sigma$ denote the sum-of-divisors function, so that $\sigma(n) = \sum_{d|n} d$. Do the above problem with $\sigma$ instead of $\varphi$.

3. Let $s(n) = \sigma(n) - n$, the sum-of-proper-divisors function. Show that for each positive integer $m$, the set $\{n : m \mid n,\ m \nmid s(n)\}$ has asymptotic density 0.

4. Let $\lambda(n)$ denote the universal exponent of the group $(\mathbb{Z}/n\mathbb{Z})^\times$. That is, $\lambda(n)$ is the least positive integer such that $a^{\lambda(n)} \equiv 1 \pmod{n}$ for all integers $a$ coprime to $n$. (It can also be defined as the order of the largest cyclic subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$.) There is a formula for $\lambda(n)$ that is similar to the one for $\varphi(n)$. In particular, for a prime power $p^a$, we have $\lambda(p^a) = \varphi(p^a)$ unless $p = 2, a \geq 3$, in which case, $\lambda(2^a) = 2^{a-2}$. And if $n$ has the prime factorization $p_1^{a_1} \dots p_k^{a_k}$ with distinct primes $p_1, \dots, p_k$, then

$$\lambda(n) = \operatorname{lcm}\{\lambda(p_1^{a_1}), \dots, \lambda(p_k^{a_k})\}.$$

   For example, $\lambda(1000) = \operatorname{lcm}\{2, 5^2 \cdot 4\} = 100$, and $\lambda(1001) = \operatorname{lcm}\{6, 10, 12\} = 60$. Show that the set of odd numbers $n$ with

$$2^{\lambda(n)/2} \not\equiv 1 \pmod{n}$$

   has asymptotic density 0. (Hint: If $p \equiv 1 \pmod 8$, then 2 is a quadratic residue for $p$.)

5. Let

$$A = \{n : n \mid 2^k - 1 \text{ for some positive integer } k\},$$
$$B = \{n : n \mid 2^k + 1 \text{ for some positive integer } k\}.$$

   Show that $A$ has asymptotic density 1/2 and $B$ has asymptotic density 0. (Hint: If $p \equiv 7 \pmod 8$, then 2 is a quadratic residue for $p$.)