Supplementary homework problems, due May 11, 2009

1. Disprove: If $f(x)$ is a non-constant, monic irreducible polynomial in $\mathbb{Z}[x]$, then $f(n)$ is prime for infinitely many integers n.

2. Formulate a conjecture about which non-constant polynomials in $\mathbb{Z}[x]$ are prime for infinitely many integer arguments.

3. We say a positive integer s is *squarefull* (or *powerful*) if $p^2 \mid s$ for each prime $p \mid s$. The sequence of squarefull integers is $1, 4, 8, 9, \ldots$. Every power is squarefull, but not converselty; for example, 72 is squarefull.

   (a) Show that every squarefull number has a (not necessarily unique) representation as a square times a cube. Use this to show that the sum of the reciprocals of the squarefull numbers is finite.

   (b) For a natural number n, let $s(n)$ denote the largest squarefull divisor of n. Show that for each squarefull number s, the set $\{n : s(n) = s\}$ has a positive asymptotic density D_s. (For example, $D_1 = 6/\pi^2$, see problem 2a in the text.)

   (c) Show that $\sum_{s \text{ squarefull}} D_s = 1$.

   (d) Use this to do problem 3a in the text.