

Winter 2019 Math 106
Topics in Applied Mathematics
Data-driven Uncertainty Quantification

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Lecture 12: Brownian Motion

12.1 Brownian Motion

Definition. Brownian motion (or **Wiener process**) is a stochastic process $w(\omega, t), \omega \in \Omega, 0 \leq t \leq 1$, that satisfies the following four axioms:

1. $w(\omega, 0) = 0$ for all ω .
2. For each ω , $w(\omega, t)$ is a *continuous function* of t .
3. For each $0 \leq s \leq t$, $w(\omega, t) - w(\omega, s)$ is a Gaussian variable with mean zero and variance $t - s$.
4. $w(\omega, t)$ has independent increments; i.e., if $0 \leq t_1 < t_2 < \dots < t_n$ then $w(\omega, t_i) - w(\omega, t_{i-1})$ for $i = 1, 2, \dots, n$ are independent.

Recommended reading. Theory of the Brownian movement by A Einstein.

12.1 Brownian Motion

Properties of Brownian Motion.

1. The correlation function of Brownian motion is $E[w(t_1)w(t_2)] = \min(t_1, t_2)$.
2. The Brownian path $w(\omega, t)$ for a given ω is nowhere differentiable with probability 1 with respect to t .

12.1 Brownian Motion

Properties of Brownian Motion.

1. The correlation function of Brownian motion is

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Idea of Proof. Assume $t_2 > t_1$,

$$\begin{aligned} E[w(t_1)w(t_2)] &= E[w(t_1)(w(t_2 - t_1) + w(t_1))] \\ &= E[w(t_1)(w(t_2) - w(t_1)) + E[w(t_1)w(t_1)]] \\ &= E[w(t_1)w(t_1)] = t_1 \end{aligned}$$

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2. The Brownian path $w(\omega, t)$ for a given ω is nowhere differentiable with probability 1 with respect to t .

Idea of Proof. $\frac{w(\omega, t+\Delta t) - w(\omega, t)}{\Delta t}$ is Gaussian with mean zero and variance $(\Delta t)^{-1}$.

12.1 Brownian Motion

White Noise. Although the Brownian motion does not have a derivative in the standard sense, it does have a derivative in a distribution sense

$$v(\omega, t) = \frac{dw(\omega, t)}{dt} = w'(\omega, t)$$

where $\int_{t_1}^{t_2} v(\omega, s) ds = w(\omega, t_2) - w(\omega, t_1)$. The derivative $v(\omega, t)$ is called **white noise**.

12.2 Heat Equation

We want to solve the heat equation

$$v_t = \frac{1}{2} v_{xx}, \quad v(x, 0) = \phi(x), x \in \mathbb{R}, t > 0,$$

which is a parabolic partial differential equation (PDE).

► By Fourier transform,

$$v(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{v}(k, t) dk,$$

$$v_x(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ike^{ikx} \hat{v}(k, t) dk,$$

$$v_{xx}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ik)^2 e^{ikx} \hat{v}(k, t) dk,$$

$$v_t(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \partial_t \hat{v}(k, t) dk,$$

12.2 Heat Equation

- ▶ Inserting these terms into the heat equation, we have

$$\partial_t \hat{v}(k, t) = -\frac{1}{2}k^2 \hat{v}(k, t),$$

$$\hat{v}(k, 0) = \hat{\phi}(k).$$

- ▶ The solution to the above ODE is

$$\hat{v}(k, t) = e^{-\frac{1}{2}k^2 t} \hat{\phi}(k)$$

- ▶ Thus, the solution $v(x, t)$ is given by

$$\begin{aligned} v(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{-\frac{1}{2}k^2 t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} \phi(x') dx' dk \\ &= \int_{-\infty}^{\infty} \frac{e^{\frac{-(x-x')^2}{2t}}}{\sqrt{2\pi t}} \phi(x') \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(k\sqrt{t} - i\left(\frac{x-x'}{\sqrt{t}}\right)\right)^2}}{\sqrt{2\pi}} dk \sqrt{t} dx' \end{aligned}$$

12.2 Heat Equation

► (cont'd)

$$\begin{aligned} &= \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-x')^2}{2t}}}{\sqrt{2\pi t}} \phi(x') dx' \\ &= \int_{-\infty}^{\infty} \frac{e^{-\frac{(x')^2}{2t}}}{\sqrt{2\pi t}} \phi(x + x') dx' \\ &= G * \phi \end{aligned}$$

where $G(x) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}$ is the Green function of the heat equation.

- As the Green function $G(x)$ is a probability density of a random variable w with mean zero and variance t ,

$$v(x, t) = E[\phi(x + w(\omega, t))].$$

12.3 Heat Equation by Random Walks

We want to approximate the solution of the heat equation on a grid.

- ▶ Discretize x and t into $x_i = ih$, $t^j = jk$, $i \in \mathbb{Z}$, $j \in \mathbb{N} \cup \{0\}$ with $t = nk$.
- ▶ We want to find a discrete function V_i^n that approximates $v(ih, nk) = v_i^n$.
- ▶ In Numerical Analysis, we learn that the solution V_i^n of the following difference equation

$$\frac{V_i^{n+1} - V_i^n}{k} = \frac{1}{2} \frac{V_{i+1}^n + V_{i-1}^n - 2V_i^n}{h^2}$$

converges to v_i^n as $h, k \rightarrow 0$ and $\lambda := \frac{1}{2} \frac{k}{h^2} \leq \frac{1}{2}$ (that is, the difference scheme is stable and consistent).

12.3 Heat Equation by Random Walks

- Choose $\lambda = 1/2$, that is, $k = h^2$. Then the difference equation becomes

$$V_i^{n+1} = \frac{1}{2} (V_{i+1}^n + V_{i-1}^n).$$

- By iterating backward in time and using the notation $V_i^0 = \phi(ih)$, we have

$$\begin{aligned} V_i^n &= \frac{1}{2} V_{i+1}^{n-1} + \frac{1}{2} V_{i-1}^{n-1} \\ &= \frac{1}{4} V_{i-2}^{n-2} + \frac{2}{4} V_i^{n-2} + \frac{1}{4} V_{i+2}^{n-2} \\ &\quad \vdots \\ &= \sum_{j=0}^n C_{j,n} \phi((-n + 2j + i)h) \end{aligned}$$

where $C_{j,n} = \frac{1}{2^n} \binom{n}{j}$.

12.3 Heat Equation by Random Walks

- ▶ Let $\eta_k, k = 1, 2, 3, \dots, n$ be a random walk

$$\eta_k = \begin{cases} h & \text{probability } 1/2, \\ -h & \text{probability } 1/2 \end{cases}$$

- ▶ $C_{j,n} = Pr(\sum_{k=1}^n \eta_k = (-n + 2j)h)$.
- ▶ Using the Central limit theorem, $\sum_{k=1}^n \eta_k$ converges to a Gaussian variable with mean 0 and variance $nh^2 = nk = t$ as $n \rightarrow \infty$.
- ▶ Thus,

$$C_{j,n} = Pr\left(\sum_{k=1}^n \eta_k = (-n + 2j)h\right) \sim \frac{e^{-(x')^2/2t}}{\sqrt{2\pi t}} 2h$$

where $x' = (-n + 2j)h$.

- ▶ Finally,

$$V_i^n \rightarrow \int_{-\infty}^{\infty} \frac{e^{-(x-x')^2/2t}}{\sqrt{2\pi t}} \phi(x') dx'$$

as $n \rightarrow \infty$.

12.4 Wiener Measure

- ▶ We showed that the solution of the heat equation $v(x, t)$ can be written as

$$v(x, t) = E[\phi(x + w(\omega, t))]$$

where $\phi(x)$ is the initial value.

- ▶ The expectation is on the sample space, the space of Brownian motions. What is the probability with respect to $w(\omega, t)$? How do we define a probability distribution on $w(\omega, t)$?
- ▶ The difficulty is that the Brownian motion is in an infinite-dimensional space.

12.4 Wiener Measure

- ▶ We consider the space of continuous functions $u(t)$ such that $u(0) = 0$. This is our sample space Ω .
- ▶ Pick an instant in time, say t_1 , and associate with this instant a window of values $(a_1, b_1]$, where $-\infty \leq a_1, b_1 \leq \infty$.
- ▶ Consider a subset of all continuous functions that pass through this window and denote it by C_1 (called a cylinder set).
- ▶ For every instant and every window, we can define a corresponding cylinder set, i.e., C_i is the subset of all continuous functions that pass through the windows $(a_i, b_i]$ at the instant t_i .
- ▶ Consider two cylinder sets, C_1 and C_2 . Then $C_1 \cap C_2$ is the set of functions that pass through both windows. Similarly, $C_1 \cup C_2$ is the set of functions that pass through either C_1 or C_2 .
- ▶ This forms an algebra (closed under finite disjoint unions, intersections, and complements).

12.4 Wiener Measure

- ▶ The probability measure of C_1 is defined as

$$Pr(C_1) = \int_{a_1}^{b_1} \frac{e^{-s_1^2/2t_1}}{\sqrt{2\pi t_1}} ds_1.$$

- ▶ There exists a σ -algebra and a probability measure dW (Wiener measure) that extends the probability on the cylinder sets.

Example.

$$\begin{aligned} v(x, t) &= E[\phi(x + w(\omega, t))] = \int \phi(x + w(\omega, t)) dW \\ &= \int_{-\infty}^{\infty} \phi(x + x') \frac{e^{-(x')^2/2t}}{\sqrt{2\pi t}} dx' \end{aligned}$$

12.4 Wiener Measure

Example. $\int w^2(\omega, 1)dW = \int_{-\infty}^{\infty} u^2 \frac{e^{-u^2/2}}{\sqrt{2\pi}} = 1$

Example. Assume that we can extend Fubini's theorem, that is, we can change the order of integrations. We want to find the expected value of $\int_0^1 w^2(\omega, s)ds$ for the Brownian motion w .

$$\begin{aligned} E\left[\int_0^1 w^2(\omega, s)ds\right] &= \int_0^1 w^2(\omega, s)dsdW \\ &= \int_0^1 ds \int dW w^2(\omega, s) = \int_0^1 sds = \frac{1}{2}. \end{aligned}$$

12.4 Wiener Measure

Example. Find the expected value of $w^2(\omega, 1/2)w^2(\omega, 1)$.

$$\int w^2(\omega, 1/2)w^2(\omega, 1)dW = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2(x+y)^2 \frac{e^{-x^2-y^2}}{\pi} dx dy = 1.$$

12.5 Heat Equation with Potential

Now we consider the following heat equation with a potential $U(x)$

$$v_t = \frac{1}{2} v_{xx} + U(x)v, \quad v(x, 0) = \phi(x).$$

- For $\lambda = \frac{1}{2} \frac{k}{h^2} \leq \frac{1}{2}$, the solution to the following difference equation converges to the solution of the original equation (a good exercise for numerical analysis)

$$\frac{V_i^{n+1} - V_i^n}{k} = \frac{1}{2} \frac{V_{i-1}^n + V_{i+1}^n - 2V_i^n}{h^2} + \frac{1}{2} (U_{i-1} V_{i-1}^n + U_{i+1} V_{i+1}^n),$$

where $U_i = U(ih)$.

12.5 Heat Equation with Potential

- ▶ For $\lambda = 1/2$,

$$\begin{aligned}V_i^{n+1} &= \frac{1}{2}(V_{i-1}^n - V_{i+1}^n) + \frac{k}{2}(U_{i+1}V_{i+1}^n + U_{i-1}V_{i-1}^n) \\&= \frac{1}{2}(1 + kU_{i+1})V_{i+1}^n + \frac{1}{2}(1 + kU_{i-1})V_{i-1}^n.\end{aligned}$$

- ▶ By induction, we have

$$V_i^n = \sum_{l_1=\pm 1, \dots, l_n=\pm 1} \frac{1}{2^n} (1+kU_{i+l_1}) \cdots (1+kU_{i+l_1+\dots+l_n}) V_{i+l_1+\dots+l_n}^0$$

- ▶ Let $\eta_k, k = 1, 2, 3, \dots, n$ be a random walk

$$\eta_k = \begin{cases} h & \text{probability } 1/2, \\ -h & \text{probability } 1/2 \end{cases}$$

- ▶ $Pr(\eta_1 = l_1 h, \dots, \eta_n = l_n h) = \frac{1}{2^n}$

12.5 Heat Equation with Potential

- ▶ A probabilistic interpretation of the solution V_i^n is

$$V_i^n = E_{\text{paths}} \{ \prod_{m=1}^n (1 + kU(ih + \eta_1 + \cdots + \eta_m)) \\ \times \phi(ih + \eta_1 + \cdots + \eta_n) \}$$

- ▶ Let $\tilde{w}(s)$ be the path connecting η_i linearly for $0 \leq s \leq t$. Then we have

$$V_i^n = E_{\text{broken line paths}} \{ \prod_{m=1}^n (1 + kU(ih + \tilde{w}(s_m))) \\ \times \phi(ih + \tilde{w}(t)) \}$$

where $s_m = mk$.

- ▶ For $k|U| < 1/2$, $(1 + kU) = \exp(kU + \epsilon)$ where $|\epsilon| \leq Ck^2$.
- ▶ $\prod_{m=1}^n (1 + kU(ih + \tilde{w}(s_m))) = \exp(k \sum_{m=1}^n U(ih + \tilde{w}(s_m)) + \epsilon')$, where $|\epsilon'| \leq nCk^2 = Ctk$.

12.5 Heat Equation with Potential

- ▶ $V_i^n = E_{\text{broken line paths}} \left\{ e^{\int_0^t U(ih + \tilde{w}(s)) ds} \phi(ih + \tilde{w}(t)) \right\} + \text{small terms.}$
- ▶ As h and k tend to zero, the broken line paths $ih + \tilde{w}(s)$ look more and more like Brownian motion paths $x + w(s)$, so in the limit,

$$\begin{aligned} v(x, t) &= E_{\text{all Brownian motion paths}} \left\{ e^{\int_0^t U(x + w(s)) ds} \phi(x + w(t)) \right\} \\ &= \int dW e^{\int_0^t U(x + w(s)) ds} \phi(x + w(t)), \end{aligned}$$

the **Feynman-Kac** formula!

12.5 Heat Equation with Potential

Feynman diagrams.. We introduce an ϵ , a small parameter, in front of the potential U . After expanding in a Taylor series of ϵ ,

$$e^{\int_0^t \epsilon U(x+w(s)) ds} = 1 + \epsilon \int_0^t U(x+w(s)) ds + \frac{1}{2} \epsilon^2 \left(\int_0^t U(x+w(s)) ds \right)^2 + \dots$$

- ▶ constant term $T_0 = \int dW \phi(x + w(t))$
 $= \int_{-\infty}^{\infty} \frac{e^{-(x-z)^2/2t}}{\sqrt{2\pi t}} \phi(z) dz = \int_{-\infty}^{\infty} K(x-z, t) \phi(z) dz$ with the **vacuum propagator** $K(z, s) = \frac{1}{\sqrt{2\pi s}} e^{-z^2/2s}$
- ▶ ϵ -order term T_1

$$\begin{aligned} T_1 &= \epsilon \int dW \int_0^t U(x+w(s)) \phi(x+w(t)) ds \\ &= \epsilon \int_0^t ds \int dW U(x+w(s)) \phi(x+w(t)). \end{aligned}$$

12.5 Heat Equation with Potential

- ▶ T_1 cont'd

$$T_1 = \epsilon \int_0^t ds \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 K(z_1 - x, s) \cdot U(z_1) K(z_2, t - s) \phi(z_1 + z_2)$$

- ▶ Similarly (straightforward but not easy), the ϵ^2 term

$$\begin{aligned} T_2 = \epsilon^2 \int_0^t dt_2 \int_0^{t_2} dt_1 \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \int_{-\infty}^{\infty} & \\ \cdot K(z_1 - x, t_1) U(z_1) K(z_2, t_2 - t_1) U(z_1 + z_2) & \\ \cdot K(z_3, t - t_2) \phi(z_1 + z_2 + z_3). & \end{aligned}$$