Winter 2019 Math 106
Topics in Applied Mathematics
Data-driven Uncertainty Quantification

Yoonsang Lee (yoonsang.lee@dartmouth.edu)

Lecture 15: Karhunen-Loeve expansion

15.1 Integral equation

Karhune-Loeve expansion provides a finite dimensional approximation of a stochastic process.
In this lecture note, we consider a real-valued stochastic process.

- For $t \in T$, let $u(\omega, t)$ be a real-valued stochastic process with a finite covariance $R(t_1, t_2)$.
- $u(\omega, t)$ has the following decomposition

$$u(\omega, t) = m(t) + \tilde{u}(\omega, t)$$

  with $E[\tilde{u}(\omega, t)] = 0$.
- Assume that **we know** $m(t)$ **and** $R(t_1, t_2)$.
- Before we continue, we need some facts from integral equations

15.1 Integral equation

Fredholm integral equations of the second kind

$$\int_T R(s_1, s_2)\psi(s_2)ds_2 = \lambda\psi(s_1)$$

- The above equation is a homogeneous Fredholm integral equation of the second kind.
- This is an eigenvalue problem.
- As the kernel, or the covariance function $R(t_1, t_2)$ is bounded, symmetric, and positive definite, the integral equation has eigenfunctions $\{\psi_i\}$ such that
  1. The set $\{\psi_i\}$ of eigenfunctions is orthogonal and complete.
  2. For each eigenvalue λ_k, there correspond at most a finite number of linearly independent eigenfunctions.
  3. There are at most a countably infinite set of eigenvalues.
  4. The eigenvalues are all positive real numbers.
  5. The kernel admits of the following uniformly convergent expansion

$$R(s_1, s_2) = \sum_{i}^{\infty} \lambda_k \psi_i(s_1)\psi_i(s_2).$$

15.2 Karhunen-Loeve expansion

Karhunen-Loeve expansion of $u(\omega, t)$

$$u(\omega, t) = \sum_{i}^{\infty} \xi_i(\omega)\sqrt{\lambda_i}\psi_i(t)$$

where $\{\psi_i\}$ are orthonormal functions, λ_i are constant, and $E[\xi_i] =$ and $E[\xi_i\xi_j] = 0$ for $i \neq j$.

- The process is not necessarily stationary; we assume that we know only the mean and the covariance function.
- The set of ψ_i is the eigenfunctions of the following eigenvalue problem $\int_T R(s_1, s_2)\psi(s_2)ds_2 = \lambda\psi(s_1)$ where λ_i is the corresponding eigenvalue of ψ_i.
- $\xi(\omega) = \frac{1}{\sqrt{\lambda_i}} \int_T u(\omega, s)\psi_i(s)ds$.
- A finite dimensional approximation of u is given by

$$u(\omega, t) \approx \sum_{i}^{N} \xi_i(\omega)\sqrt{\lambda_i}\psi_i(t).$$

15.2 Karhunen-Loeve expansion

Example (uncorrelated process). $R(t, s) = \delta(t - s)$ (white noise)
Any orthogonal functions are eigenfunctions with $\lambda_i = 1$ for all i.
That is, there is no decay in the magnitude of λ_i.

15.2 Karhunen-Loeve expansion

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Example (fully correlated process). $R(t, s) = 1$
There is only one eigenfunction $\psi = 1$ with $\lambda = 1$. Thus, we need to specify only one random variable ξ.

15.2 Karhunen-Loeve expansion

Example (exponential covariance function).
For $t \in T = [-b, b]$, let $R(t,s) = \exp(-|t-s|/a)$ for $a > 0$. a is called **correlation length**. Note that this is a stationary process in the wild sense.
The eigenvalues are

$$\lambda_i = \begin{cases} \frac{2a}{1+a^2 w_i^2} & \text{i: even} \\ \frac{2a}{1+a^2 v_i^2} & \text{i: odd} \end{cases}$$

and the corresponding eigenfunctions are

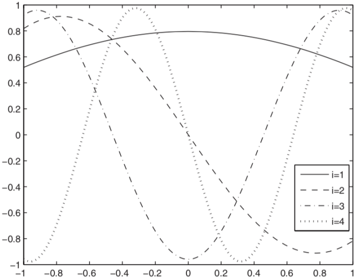
$$\psi_i = \begin{cases} \frac{\sin(w_i t)}{\sqrt{b - \frac{\sin(2w_i b)}{2w_i}}} & \text{i: even} \\ \frac{\cos(w_i t)}{\sqrt{b - \frac{\sin(2v_i b)}{2v_i}}} & \text{i: odd} \end{cases}$$

where w_i and v_i are the solutions of the transcendental equations

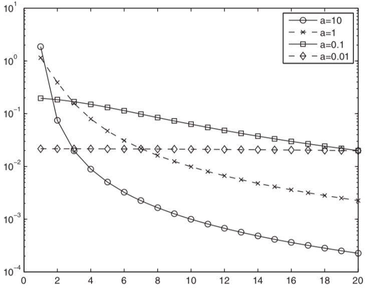
$$\begin{cases} aw + \tan(wb) = 0 \\ 1 - av + \tan(vb) = 0 \end{cases}$$

15.2 Karhunen-Loeve expansion

Example (exponential covariance function).



eigenfunctions



eigenvalues

15.3 Properties of Karhunen-Loeve expansion

Claim. A finite-term Karhunen-Loeve expansion is optimal in terms of the mean-square error.

Claim. The random variable coefficients ξ_i satisfies

$$E[\xi_i] = 0, \qquad E[\xi_i \xi_j] = 0 \quad i \neq j$$

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Proof.

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Homework

Go to https://archive.siam.org/meetings/uq18/invited.php and view a talk of your choice. Make a one-page report.

SIAM Conference on Uncertainty Quantification 2018 invited talks

- Merlise A. Clyde, Duke University, USA
  Model Uncertainty and Uncertainty Quantification
- Omar Ghattas, The University of Texas at Austin, USA
  Scalable Algorithms for PDE-constrained Optimization Under Uncertainty
- Michael I. Jordan, University of California, Berkeley, USA
  On Gradient-Based Optimization: Accelerated, Stochastic and Nonconvex
- Fabio Nobile, Ecole Polytechnique Fédérale de Lausanne (EPFL), Switzerland
  Multi-level and Multi-index Monte Carlo Methods in Practice
- Sebastian Reich, Universität Potsdam, Germany and University of Reading, United Kingdom
  Data Assimilation and Uncertainty Quantification — A Lagrangian Interacting Particle Perspective
- Johannes O. Royset, Naval Postgraduate School, USA
  Good and Bad Uncertainty: Consequences in UQ and Design
- Ian H. Sloan, The University of New South Wales, Australia
  A Contemporary View of High-dimensional Quasi Monte Carlo
- Bin Yu, University of California, Berkeley, USA
  Three Principles of Data Science: Predictability, Stability, and Computability