Math 106 Data-driven Uncertainty Quantification
Homework 01

Winter 2021

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Due Jan 24, 2021 11:59 pm (EDT)

Starred problems are for math-major students. Non-math major students are welcome to try starred problems.

1. * For X$\sim$ Bernoulli distribution with p, specify a probability space $(\Omega, \mathcal{B}, \mu)$ where the sample space is continuous, say $\Omega = [0, 1]$.

2. Let X_1 and X_2 are two independent uniform distributions on $(0, 1)$.

   (a) Find the density of $Y_1 = X_1 + X_2$

   (b) Find the density of $Y_2 = X_1/X_2$

   (c) Find the density of $Y_3 = X_1^2$

   (d) Find the density of $Y_4 = \max(X_1, X_2)$

3. Let X and Y be two random variables with $E[Y] = m$ and $E[Y^2] < \infty$.

   (a) Show that the constant c that minimizes $E[(Y - c)^2]$ is $c = m$.

   (b) Show that the random variable $f(X)$ that minimizes $E[(Y - f(X))^2|X]$ is
   $$f(X) = E[Y|X].$$

   (c) Show that the random variable $f(X)$ that minimizes $E[(Y - f(X))^2]$ is also
   $$f(X) = E[Y|X].$$

4. Will you consider a coin asymmetric if after 1000 coin tosses the number of heads is equal to 550?

5. For two random variables X and Y, show that
   $$H(X|Y) \leq H(X)$$

6. (a) State the Jensen's inequality. (b)* (for math-majors) prove the inequality.

7. * Let X and Y be independent standard normal. Construct a new (standard normal) random variable Z using X and Y so that the correlation between X and Z is ρ. What is the correlation between Y and Z?

8. - Write a code that generates a sample of n values from the standard normal distribution $N(0,1)$. n is an input parameter of the code.
   - Draw a histogram of the sample.
   - Draw the Gaussian fit to the sample statistics. That is, draw the Gaussian density with the same mean and variance of the sample.
   - Draw a histogram of $y_i = x_i^2$ where x_i is a sample from the standard normal distribution.

9. - Write a code that draws a sample of n values of the uniform distribution on $[0,1]$. n is an input parameter of the code.
   - Use a transformation of random variables to generate samples from the Cauchy density $p(x) = \frac{1}{\pi(1+x^2)}$.
   - Draw a histogram of the sample.
   - Calculate the mean. Plot the mean as a function of n.

10. - Draw n values of the standard normal random variable, X.
    - When $Y = X^2$, calculate $D(X,Y)$ using the sample. If you use a histogram in a sense, change the number of bins and check the change of the relative entropy.
    - Compare the relative entropy with an analytic solution (it is okay to use computer to finish your analytic formula of the relative entropy).