Winter 2021 Math 106
Topics in Applied Mathematics
Data-driven Uncertainty Quantification

Yoonsang Lee (yoonsang.lee@dartmouth.edu)

Lecture 6: Bayesian Inference

6.1 Bayesian inference

In the estimation problem of a variable θ of interest using a sample a sample $\{X_i\}$,
the conditional probability density of θ for the given sample $\{X_i\}$ is given by

$$p(\theta|\{X_i\}) = \frac{p(\theta)p(\{X_i\}|\theta)}{\int p(\theta)p(\{X_i\}|\theta)d\theta}$$

from the Bayes' theorem.

- $p(\theta)$ is a prior density of θ.
- $p(\{X_i\}|\theta)$ is the likelihood of $\{X_i\}$.
- The denominator is a normalization constant.

6.1 Bayesian inference

In Lecture 4, we discussed a parametric inference problem using a parameter θ and a sample $\{X_i\}$.

- Likelihood $\mathcal{L}_n(\theta) = \Pi_i^n p(X_i; \theta)$.
- The likelihood is not a probability density of θ.
- θ is a fixed value and we make probability statements only for the random variables related to the sample for an increasing sample size.

In Bayesian inference,

- We make probability statements about θ, that is, θ is a random variable.
- The probability describes degree of belief.
- For example, "the probability that it will rain tomorrow is .35"

6.1 Bayesian inference

What do we do with the posterior density?

- For a point estimate, we can use the mean of mode of the posterior
- We can also obtain a Bayesian interval estimate C

$$\mu(\theta \in C | \{X_i\}) = \int_C p(\theta | \{X_i\}) d\theta = 1 - \alpha.$$

Here, we assume that θ is a random variable and $\{X_i\}$ is fixed.

6.2 Priors

- If we assume a constant for the prior, that is, a uniform density, the mode of the posterior is equal to the maximum likelihood estimator (MLE) because

  $$p(\theta|\{X_i\}) \approx p(\{X_i\}|\theta).$$

  Thus, MLE is related to the Bayesian estimator.

- However, this does not always hold; if $\theta \in \mathbb{R}$, there is no uniform density on $\mathbb{R}$ because

  $$\int_{\mathbb{R}} c dx = \infty.$$

  for any constant $c > 0$.

6.2 Priors

- A constant prior is **not** transformation invariant.
  Let's assume a uniform prior density for $\theta \in (0, 1)$ because of lack of any prior information. For a transformation of θ, $\psi = \ln(\theta/(1 - \theta))$, we also have no prior information and we may assume a uniform prior density for ψ.
  It is a straightforward exercise to check that the density of ψ is

  $$p(\psi) = \frac{e^{\psi}}{(1 + e^{\psi})^2}$$

  if we assume a uniform density for θ.

Exercise. Let $X_1, X_2, ..., X_n$ be IID of $N(\theta, \sigma^2)$ where θ is unknown and σ is known. Suppose we take as a prior θ is $N(a_{prior}, b^2)$ where a_{prior} and b are known constants.

- The posterior is Gaussian, that is, $p(\theta|\{X_i\}) = \phi(x; a_{post}, b_{post}^2)$ where ϕ is a Gaussian density.
- The posterior mean and variance are

$$a_{post} = k\left(\frac{1}{n}\sum_i X_i\right) + (1-k)a_{prior} = a_{prior} + k\left(\frac{1}{n}\sum_i X_i - a_{prior}\right)$$

where

$$k = \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{b^2}}$$

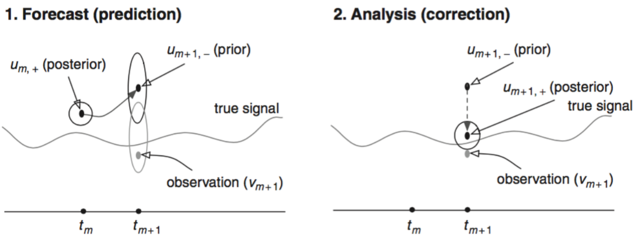
and

$$b_{post}^2 = \frac{b^2\sigma^2/n}{b^2 + \sigma^2/n}$$

6.3 Kalman Filtering

- Kalman filter was co-invented and developed by R.E. Kalman (National Medal of Science 2009).
- Kalman filter is also known as linear quadratic estimation (LQE).
- Kalman filter uses a series of measurements observed over time to estimate unknown variables.
- Kalman filter estimate the conditional density of unknown variables at each time when measurements are available.

6.3 Kalman Filtering



$u_{m+1,post}$: posterior mean at the $m+1$-th step.
$u_{m+1,prior}$: prior mean at the $m+1$-th step.
v_{m+1}: observation at the $m+1$-th step.

$$u_{m+1,post} = u_{m+1,prior} + K(v_{m+1} - u_{m+1,prior})$$

where K is the Kalman gain

$$K = \frac{\sigma^2_{m+1,prior}}{\sigma^2_{obs} + \sigma^2_{m+1,prior}}$$