Winter 2021 Math 106
Topics in Applied Mathematics
Data-driven Uncertainty Quantification

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Lecture 12: Brownian Motion

12.1 Brownian Motion

Definition. **Brownian motion** (or **Wiener process**) is a stochastic process $w(\omega, t), \omega \in \Omega, 0 \leq t \leq 1$, that satisfies the following four axioms:

1. $w(\omega, 0) = 0$ for all ω.
2. For each ω, $w(\omega, t)$ is a *continuous function* of t.
3. For each $0 \leq s \leq t$, $w(\omega, t) - w(\omega, s)$ is a Gaussian variable with mean zero and variance $t - s$.
4. $w(\omega, t)$ has independent increments; i.e., if $0 \leq t_1 < t_2 < \cdots < t_n$ then $w(\omega, t_i) - w(\omega, t_{i-1})$ for $i = 1, 2, ..., n$ are independent.

Recommended reading. Theory of the Brownian movement by A Einstein.

12.1 Brownian Motion

Properties of Brownian Motion.

1. The correlation function of Brownian motion is $E[w(t_1)w(t_2)] = \min(t_1, t_2)$.

2. The Brownian path $w(\omega, t)$ for a given ω is nowhere differentiable with probability 1 with respect to t.

12.1 Brownian Motion

Properties of Brownian Motion.

1. The correlation function of Brownian motion is $E[w(t_1)w(t_2)] = \min(t_1, t_2)$.
   Idea of Proof. Assume $t_2 > t_1$,

$$E[w(t_1)w(t_2)] = E[w(t_1)\,(w(t_2 - t_1) + w(t_1))]$$

$$= E[w(t_1)\,(w(t_2) - w(t_1)) + E[w(t_1)w(t_1)]$$

$$= E[w(t_1)w(t_1)] = t_1$$

2. The Brownian path $w(\omega, t)$ for a given ω is nowhere differentiable with probability 1 with respect to t.

12.1 Brownian Motion

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2. The Brownian path $w(\omega, t)$ for a given ω is nowhere differentiable with probability 1 with respect to t.
   Idea of Proof. $\frac{w(\omega,t+\Delta t)-w(\omega,t)}{\Delta t}$ is Gaussian with mean zero and variance $(\Delta t)^{-1}$.

12.1 Brownian Motion

White Noise. Although the Brownian motion does not have a derivative in the standard sense, it does have a derivative in a distribution sense

$$v(\omega, t) = \frac{dw(\omega, t)}{dt} = w'(\omega, t)$$

where $\int_{t_1}^{t_2} v(\omega, s)ds = w(\omega, t_2) - w(\omega, t_1)$. The derivative $v(\omega, t)$ is called **white noise**.

12.2 Heat Equation

We want to solve the heat equation

$$v_t = \frac{1}{2} v_{xx}, \quad v(x,0) = \phi(x), x \in \mathbb{R}, t > 0,$$

which is a parabolic partial differential equation (PDE).

- By Fourier transform,

$$v(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \hat{v}(k,t) dk,$$

$$v_x(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ike^{ikx} \hat{v}(k,t) dk,$$

$$v_{xx}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ik)^2 e^{ikx} \hat{v}(k,t) dk,$$

$$v_t(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \partial_t \hat{v}(k,t) dk,$$

12.2 Heat Equation

- Inserting these terms into the heat equation, we have

$$\partial_t \hat{v}(k,t) = -\frac{1}{2}k^2 \hat{v}(k,t),$$

$$\hat{v}(k,0) = \hat{\phi}(k).$$

- The solution to the above ODE is

$$\hat{v}(k,t) = e^{-\frac{1}{2}k^2 t}\hat{\phi}(k)$$

- Thus, the solution $v(x,t)$ is given by

$$v(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} e^{-\frac{1}{2}k^2 t} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx'}\phi(x')dx'dk$$

$$= \int_{-\infty}^{\infty} \frac{e^{\frac{-(x-x')^2}{2t}}}{\sqrt{2\pi t}}\phi(x')\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(k\sqrt{t}-i(\frac{x-x'}{\sqrt{t}})\right)^2}}{\sqrt{2\pi}} dk\sqrt{t}dx'$$

12.2 Heat Equation

- (cont'd)

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-x')^2}{2t}}}{\sqrt{2\pi t}} \phi(x')dx'$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{(x')^2}{2t}}}{\sqrt{2\pi t}} \phi(x+x')dx'$$

$$= G * \phi$$

  where $G(x) = \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}$ is the Green function of the heat equation.

- As the Green function $G(x)$ is a probability density of a normal random variable w with mean zero and variance t,

$$v(x,t) = E[\phi(x + w(\omega, t))].$$

12.3 Heat Equation by Random Walks

We want to approximate the solution of the heat equation on a grid.

- Discretize x and t into $x_i = ih$, $t^j = jk$, $i \in \mathbb{Z}, j \in \mathbb{N} \cup \{0\}$ with $t = nk$.
- We want to find a discrete function V_i^n that approximates $v(ih, nk) = v_i^n$.
- In Numerical Analysis, we learn that the solution V_i^n of the following difference equation

$$\frac{V_i^{n+1} - V_i^n}{k} = \frac{1}{2}\frac{V_{i+1}^n + V_{i-1}^n - 2V_i^n}{h^2}$$

converges to v_i^n as $h, k \to 0$ and $\lambda := \frac{1}{2}\frac{k}{h^2} \leq \frac{1}{2}$ (that is, the difference scheme is stable and consistent).

12.3 Heat Equation by Random Walks

- Choose $\lambda = 1/2$, that is, $k = h^2$. Then the difference equation becomes

$$V_i^{n+1} = \frac{1}{2}\left(V_{i+1}^n + V_{i-1}^n\right).$$

- By iterating backward in time and using the notation $V_i^0 = \phi(ih)$, we have

$$\begin{aligned} V_i^n &= \frac{1}{2}V_{i+1}^{n-1} + \frac{1}{2}V_{i-1}^{n-1} \\ &= \frac{1}{4}V_{i-2}^{n-2} + \frac{2}{4}V_i^{n-2} + \frac{1}{4}V_{i+2}^{n-2} \\ &\vdots \\ &= \sum_{j=0}^{n} C_{n,j}\phi((-n+2j+i)h) \end{aligned}$$

where $C_{n,j} = \frac{1}{2^n}\begin{pmatrix} n \\ j \end{pmatrix}$.

12.3 Heat Equation by Random Walks

- Let $\eta_k, k = 1, 2, 3, ..., n$ be a random walk

$$\eta_k = \begin{cases} h & \text{probability } 1/2, \\ -h & \text{probability } 1/2 \end{cases}$$

- $C_{n,j} = Pr\left(\sum_{k=1}^{n} \eta_k = (-n + 2j)h\right).$
- Using the Central limit theorem, $\sum_k^n \eta_k$ converges to a Gaussian variable with mean 0 and variance $nh^2 = nk = t$ as $n \to \infty$.
- Thus,

$$C_{n,j} = Pr\left(\sum_{k=1}^{n} \eta_k = (-n + 2j)h\right) \sim \frac{e^{-(x')^2/2t}}{\sqrt{2\pi t}} 2h$$

where $x' = (-n + 2j)h$.
- Finally,

$$V_i^n \to \int_{-\infty}^{\infty} \frac{e^{-(x-x')^2/2t}}{\sqrt{2\pi t}} \phi(x') dx'$$

as $n \to \infty$.

12.4 Wiener Measure

- We showed that the solution of the heat equation $v(x,t)$ can be written as

$$v(x,t) = E[\phi(x + w(\omega, t))]$$

  where $\phi(x)$ is the initial value.
- The expectation is on the sample space, the space of Brownian motions. What is the probability with respect to $w(\omega, t)$? How do we define a probability distribution on $w(\omega, t)$?
- The difficulty is that the Brownian motion is in an infinite-dimensional space.

12.4 Wiener Measure

- We consider the space of continuous functions $u(t)$ such that $u(0) = 0$. This is our sample space Ω.
- Pick an instant in time, say t_1, and associate with this instant a window of values $(a_1, b_1]$, where $-\infty \leq a_1, b_2 \leq \infty$.
- Consider a subset of all continuous functions that pass through this window and denote it by C_1 (called a cylinder set).
- For every instant and every window, we can define a corresponding cylinder set, i.e., C_i is the subset of all continuous functions that pass through the windows $(a_i, b_i]$ at the instant t_i.
- Consider two cylinder sets, C_1 and C_2. Then $C_1 \cap C_2$ is the set of functions that pass through both windows. Similarly, $C_1 \cup C_2$ is the set of functions that pass through either C_1 or C_2.
- This forms an algebra (closed under finite disjoint unions, intersections, and complements).

12.4 Wiener Measure

- The probability measure of C_1 is defined as

$$Pr(C_1) = \int_{a_1}^{b_1} \frac{e^{-s_1^2/2t_1}}{\sqrt{2\pi t_1}} ds_1.$$

- There exists a σ-algebra and a probability measure dW (Wiener measure) that extends the probability on the cylinder sets.

Example.

$$v(x,t) = E[\phi(x + w(\omega,t))] = \int \phi(x + w(\omega,t))dW$$

$$= \int_{-\infty}^{\infty} \phi(x + x') \frac{e^{-(x')^2/2t}}{\sqrt{2\pi t}} dx'$$

12.4 Wiener Measure

Example. $\int w^2(\omega,1)dW = \int_{-\infty}^{\infty} u^2 \frac{e^{-u^2/2}}{\sqrt{2\pi}} = 1$

Example. Assume that we can extend Fubini's theorem, that is, we can change the order of integrations. We want to find the expected value of $\int_0^1 w^2(\omega,s)ds$ for the Brownian motion w.

$$E[\int_0^1 w^2(\omega,s)ds] = \int_0^1 w^2(\omega,s)dsdW$$

$$= \int_0^1 ds \int w^2(\omega,s)dW = \int_0^1 sds = \frac{1}{2}.$$

12.4 Wiener Measure

Example. Find the expected value of $w^2(\omega, 1/2)w^2(\omega, 1)$.

$$\int w^2(\omega, 1/2)w^2(\omega, 1)dW = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2(x+y)^2\frac{e^{-x^2-y^2}}{\pi}dxdy = 1.$$

12.5 Heat Equation with Potential

Now we consider the following heat equation with a potential $U(x)$

$$v_t = \frac{1}{2} v_{xx} + U(x) v, \quad v(x, 0) = \phi(x).$$

- For $\lambda = \frac{1}{2}\frac{k}{h^2} \leq \frac{1}{2}$, the solution to the following difference equation converges to the solution of the original equation (a good exercise for numerical analysis)

$$\frac{V_i^{n+1} - V_i^n}{k} = \frac{1}{2}\frac{V_{i-1}^n + V_{i+1}^n - 2V_i^n}{h^2} + \frac{1}{2}\left(U_{i-1}V_{i-1}^n + U_{i+1}V_{i+1}^n\right),$$

where $U_i = U(ih)$.

12.5 Heat Equation with Potential

- For $\lambda = 1/2$,

$$V_i^{n+1} = \frac{1}{2}(V_{i-1}^n - V_{i+1}^n) + \frac{k}{2}(U_{i+1}V_{i+1}^n + U_{i-1}V_{i-1}^n)$$

$$= \frac{1}{2}(1 + kU_{i+1})V_{i+1}^n + \frac{1}{2}(1 + kU_{i-1})V_{i-1}^n.$$

- By induction, we have

$$V_i^n = \sum_{l_1=\pm 1,\ldots,l_n=\pm 1} \frac{1}{2^n}(1+kU_{i+l_1})\cdots(1+kU_{i+l_1+\cdots+l_n})V_{i+l_1+\cdots+l_n}^0$$

- Let $\eta_k, k = 1, 2, 3, \ldots, n$ be a random walk

$$\eta_k = \begin{cases} h & \text{probability } 1/2, \\ -h & \text{probability } 1/2 \end{cases}$$

- $Pr(\eta_1 = l_1 h, \ldots, \eta_n = l_n h) = \frac{1}{2^n}$

12.5 Heat Equation with Potential

- A probabilistic interpretation of the solution V_i^n is

$$V_i^n = E_{\forall \text{paths}} \{ \Pi_{m=1}^n (1 + kU(ih + \eta_1 + \cdots + \eta_m))$$

$$\times \phi(ih + \eta_1 + \cdots + \eta_n) \}$$

- Let $\tilde{w}(s)$ be the path connecting η_i linearly for $0 \leq s \leq t$. Then we have

$$V_i^n = E_{\forall \text{broken line paths}} \{ \Pi_{m=1}^n (1 + kU(ih + \tilde{w}(s_m)))$$

$$\times \phi(ih + \tilde{w}(t)) \}$$

  where $s_m = mk$.

- For $k|U| < 1/2$, $(1 + kU) = \exp(kU + \epsilon)$ where $|\epsilon| \leq Ck^2$.
- $\Pi_{m=1}^n (1 + kU(ih + \tilde{w}(s_m))) =$
  $\exp\left(k \sum_{m=1}^n U(ih + \tilde{w}(s_m)) + \epsilon'\right)$, where $|\epsilon'| \leq nCk^2 = Ctk$.

12.5 Heat Equation with Potential

- $V_i^n = E_{\forall \text{borken line paths}} \left\{ e^{\int_0^t U(ih+\tilde{w}(s))ds} \phi(ih + \tilde{w}(t)) \right\} +$ small terms.
- As h and k tend to zero, the broken line paths $ih + \tilde{w}(s)$ look more and more like Brownian motion paths $ih + w(s)$, so in the limit,

$$v(x,t) = E_{\text{all Brownian motion paths}} \left\{ e^{\int_0^t U(x+w(s))ds} \phi(x + w(t)) \right\}$$

$$= \int dW e^{\int_0^t U(x+w(s))ds} \phi(x + w(t)),$$

the **Feynman-Kac** formula!

12.5 Heat Equation with Potential

Feynman diagrams.. We introduce an ϵ, a small parameter, in front of the potential U. After expanding in a Taylor series of ϵ,

$$e^{\int_0^t \epsilon U(x+w(s))ds} = 1+\epsilon \int_0^t U(x+w(s))ds+\frac{1}{2}\epsilon^2 \left(\int_0^t U(x+w(s))ds\right)^2 +\cdots$$

- constant term $T_0 = \int dW \phi(x+w(t))$ $= \int_{-\infty}^{\infty} \frac{e^{-(x-z)^2/2t}}{\sqrt{2\pi t}} \phi(z)dz = \int_{-\infty}^{\infty} K(x-z,t)\phi(z)dz$ with the **vacuum propagator** $K(z,s) = \frac{1}{\sqrt{2\pi s}} e^{-z^2/2s}$
- ϵ-order term T_1

$$T_1 = \epsilon \int dW \int_0^t U(x+w(s))\phi(x+w(t))ds$$

$$= \epsilon \int_0^t ds \int dW U(x+w(s))\phi(x+w(t)).$$

12.5 Heat Equation with Potential

- T_1 cont'd

$$T_1 = \epsilon \int_0^t ds \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 K(z_1 - x, s) \cdot U(z_1) K(z_2, t-s) \phi(z_1 + z_2)$$

- Similarly (straightforward but not easy), the ϵ^2 term

$$T_2 = \epsilon^2 \int_0^t dt_2 \int_0^{t_2} dt_1 \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 \int_{-\infty}^{\infty} \cdot$$

$$\cdot K(z_1 - x, t_1) U(z_1) K(z_2, t_2 - t_1) U(z_1 + z_2)$$

$$\cdot K(z_3, t - t_2) \phi(z_1 + z_2 + z_3).$$