Winter 2021 Math 106
Topics in Applied Mathematics
Data-driven Uncertainty Quantification

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Lecture 14: Stationary Stochastic Process

14.1 Stationary Stochastic Process

Definition. $u(\omega, t) \in \mathbb{C}$ is a complex-valued stochastic process if the real and imaginary parts of u are stochastic processes.

Let $m(t) = E[u(\omega, t)]$, i.e., the mean of u at time t.

Definition. $R(t_1, t_2) = E[(u(\omega, t_1) - m(t_1))\overline{u(\omega, t_2) - m(t_2)}]$

Properties of $R(t_1, t_2)$.

1. $R(t_1, t_2) = \overline{R(t_2, t_1)}$
2. $R(t_1, t_1) \geq 0$
3. $|R(t_1, t_2)| \leq \sqrt{R(t_1, t_2)R(t_2, t_1)}$
4. For all $t_1, t_2, ..., t_n$ and all $z_1, z_2, ..., z_n \in \mathbb{C}$,

$$\sum_i^n \sum_j^n R(t_i, t_j) z_i \bar{z}_j \geq 0$$

Proof of 4. For any choice of complex numbers z_j,

$$\sum_i^n \sum_j^n R(t_i, t_j) z_i \bar{z}_j = E\left[\left|\sum_j^n (u(\omega, t_j) - m(t_j)) z_j\right|^2\right] \geq 0$$

14.1 Stationary Stochastic Process

Definition. A process is stationary in the strict sense if for any $t_1, t_2, ..., t_n$ and $T \in \mathbb{R}$ $u(t_1), u(t_2), ..., u(t_n)$ and $u(t_1 + T), u(t_2 + T), ..., u(t_n + T)$ have the same distribution.

- A stationary stochastic process in the strict sense has moments that are independent of time.
- $R(t_1 - t_2) = R(t_1, t_2)$

Properties of $R(t)$.

1. $R(t) = \overline{R(-t)}$
2. $R(0) \geq 0$
3. $|R(t)| \leq R(0)$
4. For any $t_1, t_2, ..., t_n$ and $z_1, z_2, ..., z_n \in \mathbb{C}$

$$\sum_i \sum_j R(t_i - t_j) z_i \bar{z}_j \geq 0$$

14.1 Stationary Stochastic Process

Definition. A stochastic process is stationary in the wild sense if it has a constant mean and its covariance function depends only on the different between the arguments, i.e.,

- $m(t) = m$
- $R(t_1, t_2) = R(t_1 - t_2)$

Example.

- Brownian motion is not stationary.
- White noise is stationary.
- A stationary Gaussian stochastic process in the wild sense is stationary in the strict sense.

In this course, we consider stationary processes that are stationary in the wile sense.

14.1 Stationary Stochastic Process

Pick $\xi \in \mathbb{C}$ to be a random variable and $h(t)$ a nonrandom function of time. We will consider

$$u(\omega, t) = \xi(\omega)h(t).$$

- $E[u(\omega, t)] = E[\xi]h(t)$ is constant if $h(t)$ is constant or $E[\xi] = 0$.
- Suppose $E[\xi] = 0$,

  $$R(t_1, t_2) = E[\xi h(t_1)\xi \bar{h(t_2)}] = E[\xi\bar{\xi}]h(t_1)\bar{h(t_2)}$$

  must depends only on $t_1 - t_2$.
- If $t_1 = t_2 = t$, $E[\xi\bar{\xi}]h(t)\bar{h(t)}$ must be $R(0)$, and thus $h(t)\bar{h(t)}$ is constant

  $$h(t) = Ae^{i\phi(t)}$$

14.1 Stationary Stochastic Process

- Suppose $A \neq 0$,

$$R(t_1 - t_2) = |A|^2 E[\xi\bar{\xi}] e^{i\phi(t_1) - i\phi(t_2)}$$

- Using $t_2 = t, t_1 - t_2 = T$,

$$R(T) = |A|^2 E[\xi\bar{\xi}] e^{i} \phi(t+T) - \phi(t)$$

- To satisfy $R(T) = \overline{R(-T)}$,

$$\phi(t+T) - \phi(t) = -\phi(t-T) + \phi(t)$$

$$\Rightarrow \phi(t+T) - 2\phi(t) + \phi(t-T) = 0$$

$$\Rightarrow \phi''(t) = 0 \quad \text{for all } t$$

  Thus, $\phi(t) = \lambda t + \beta$.

Conclusion. $u(\omega, t) = \xi(\omega) h(t)$ is stationary in the wild sense if $h(t) = Ce^{i\lambda t}$ and $E[\xi] = 0$. Its covariance function is $R(T) = E[\xi^2] e^{i\lambda T}$.

14.2 Covariance and Spectrum

Consider a more general form $u(\omega, t) = \xi_1(\omega)e^{i\lambda_1 t} + \xi(_2\omega)e^{i\lambda_2 t}$ with $\lambda_1 \neq \lambda_2$.

- $E[u] = E[\xi_1]e^{i\lambda_1 t} + E[\xi_2]e^{i\lambda_2 t}$, which is independent of t if $E[\xi_1] = E[\xi_2] = 0$.
- $E[(\xi_1(\omega)e^{i\lambda_1 t} + \xi(_2\omega)e^{i\lambda_2 t})\overline{(\xi_1(\omega)e^{i\lambda_1 t} + \xi(_2\omega)e^{i\lambda_2 t}})]$
  $= E[|\xi_1|^2 e^{i\lambda_1 T} + |\xi_2|^2 e^{i\lambda_2 T} + \xi_1\bar{\xi_2}e^{i\lambda_1 t_2 - i\lambda_2 t_2} + \bar{\xi_1}\xi_2 e^{i\lambda_1 t_1 - i\lambda_2 t_2}]$
  which can be stationary only if $E[\xi_1\bar{\xi_2}] = 0$.
- If $E[\xi_1\bar{\xi_2}] = 0$,

$$R(T) = E[|\xi_1|^2]e^{i\lambda_1 T} + E[|\xi_2|^2]e^{i\lambda_2 T}$$

A generalization of this says a process $u = \sum_j \xi_j e^{i\lambda_j t}$ is stationary in the wild sense if $E[\xi_j\bar{\xi_j}] = 0$ when $j \neq k$ and $E[\xi_j] = 0$.
In this case,

$$R(T) = \sum_j E[|\xi_j|^2]e^{i\lambda_j T}.$$

14.2 Covariance and Spectrum

Definition. $G(k) = \sum_{j|\lambda_j \leq k} E[|\xi_j|^2]$, the sum of expected values of the squares of the amplitudes with frequencies less than K.

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Theorem. (Khinchin)

1. If $R(T)$ is the covariance function of a stochastic process $u(\omega, t)$ stationary in the wild sense such that

$$\lim_{n \to 0} E[(u(t+h) - u(t))^2] = 0,$$

then $R(T) = \int e^{ikTdG(k)}$.

2. If a function $R(T)$ can be written as $\int e^{ikT} dG(k)$ for some nondecreasing function G, then there exists a stochastic process, stationary in the wild sense, satisfying the condition in part (1) of the theorem, that has $R(T)$ as its covariance.

14.2 Covariance and Spectrum

- If $G(k)$ is differentiable, i.e., $dG = g(k)dk$, $g(k)$ is called the **spectral density of the process**.
- That is, $R(T)$ is a Fourier transform of the spectral density
- Also, $g(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikT} R(T) dT$

Example. White noise $R(T) = \delta(T)$ and $g(k)$ is constant.

14.3 Time Series

Time series $u(\omega, t), t \in \mathbb{N}$ or $\mathbb{Z}$, a stochastic process index by a discrete set.
Assume that $E[u(t)] = 0$. Then $R(T) = E[u(t+T)\overline{u(t)}]$ has the following properties

1. $R(0) \geq 0$
2. $|R(T)| \leq R(0)$
3. $R(T) = \overline{R(-T)}$
4. $\sum_{ij} R(i-j) z_j \bar{z}_j \geq 0$.

- If $u = \xi(\omega)h(t)$, we have $h(t) = Ae^{i\phi(t)}$.
- Using $R(1) = \overline{R(-1)}$, we obtain

$$\phi(t+1) - \phi(t) = -(\phi(t-1) - \phi(t)) \bmod 2\pi \text{ for } t = 0, \pm 1, \pm 2, \dots$$

- Set $\phi(0) = \alpha$ and $\phi(0) - \phi(-1) = \lambda$. Using induction, $phi(t) = \alpha + \lambda t \bmod 2\pi$ and $h(t) = Ae^{i(\alpha + \lambda t)} = Ce^{i\lambda t}$.
- $g(k) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} R(T) e^{-iTk}$ and $R(T) = \int_{-\pi}^{\pi} e^{iTk} g(k) dk$.

14.3 Time Series

Example. We want to estimate $u(\omega, t+m), m \geq 0$ given $u(\omega, t-n), ..., u(\omega, t-1)$.

- Our estimate $\hat{u}(t+m)$ is the minimizer of

$$E[|u(t+m) - \hat{u}(t+m)|^2] \tag{1}$$

- That is,

$$\hat{u}(t+m) = E[u(t+m)|u(t-n), ..., u(t-1)] \tag{2}$$

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- That is,

$$\hat{u}(t+m) = E[u(t+m)|u(t-n), ..., u(t-1)] \tag{2}$$

Exercise. Derive (2) from (1).

14.3 Time Series

For a basis $\{\phi_i\}$ in the space of functions of $u(t-n), u(t-n+1), ..., u(t-1)$,

$$\hat{u}(t+m) = E[u(t+m)|u(t-n), ..., u(t-1)]$$

$$\approx \sum_j^n a_j \phi_j(\{u(t-n), u(t-n+1), ..., u(t-1)\})$$

A natural choice for ϕ_i is $\{u(t-n), u(t-n+1), ..., u(t-1)\}$, i.e.,

$$\boxed{\hat{u}(t+m) = \sum_j^n a_j u(t-j)}$$

a linear prediction for time series (or a autoregressive model).

14.3 Time Series

How to find a_j. Find $\{a_j\}$ that minimizes $E[|u(t+m) - \sum_j a_j u(t-j)|^2]$

$$= E[(u(t+m) - \sum_j a_j u(t-j)))\overline{(u(t+m) - \sum_k a_k u(t-k))^2})]$$

$$= E[u(t+m)\overline{u(t+m)} - \sum_k \bar{a}_k u(t+m)\overline{u(t-k)} - \sum_j a_j \overline{u(t+m)} u(t-j)$$

$$+ \sum_j \sum_k a_j \bar{a}_k u(t-j)\overline{u(t-k)}]$$

$$= R(0) - 2Re(\sum_j \bar{a}_j R(m+j)) + \sum_j \sum_k a_j \bar{a}_k R(k-j)$$

Now take a partial derivative $\frac{\partial E[|u(t+m) - \sum_j a_j u(t-j)|^2]}{\partial \bar{a}_j}$

$$= -R(m+j) + \sum_k a_k R(j-k) = 0.$$

There are n-linear equations of n unknowns, which is solvable (sure?)

Homework

1. Derive equation (2) from equation (1).
2. Numerically solve $du = -udt + dw$ up to $t = 100$. $u(0) = 10$ and use a time step $k = 0.01$. Use seed(1) in your code. Plot the covariance function of your solution.
3. Repeat 2 with $u(0) = 0$. Compare with problem 2. Discuss the results.
4. Repeat 2 with $u(0) = 0$ and $k = 1$. Compare with problem 3. Discuss the results.

Homework

5 Numerically solve the following deterministic 40-dimensional ODE with a periodic boundary condition

$$\frac{du_i}{dt} = (u_{i+1} - u_{i-2})u_{i-1} - u_i + F, i = 1, ..., N = 40$$

Use a time step $k = 0.1$ and $F = 8$. Initialize u using a Gaussian distribution with mean 0 and variance 1. Solve up to $t = 1000$. Using the solution from $t = 500$ to $t = 1000$,

(a) plot the distribution of u_1 and u_{20}. Calculate the relative entropy using the distribution of u_1 as truth.

(b) plot the covariance function of u_{20}.

(c) plot the absolute values of the Fourier transform of the covariance function of u_{20}.

6 Repeat problem 5 with $F = 6$

Homework

7 Download the stock price data of Google from the course webpage (GOOG.csv)

(a) Calculate the covariance function using data up to Dec 31, 2019.

(b) Use the covariance function to make predictions after Dec 31, 2019.

(c) Repeat the question using data from Jan 1, 2019 to Dec 31, 2019.

(d) What else can you try to improve your prediction performance? Discuss your results using the mathematical assumptions we had.