Math 108. Topics in Combinatorics.
Problem Set 1. Due on Thursday, 9/29/16.

1. Let $\pi \in \mathcal{S}_n$ be random (chosen from the uniform distribution). Fix $1 \leq k \leq n$. What is the probability that in the disjoint cycle decomposition of π, the length of the cycle containing 1 is k?

2. At the colloquium talk on 9/15/2016, Oleg Viro mentioned that "every permutation can be written as a product of two involutions." Prove it. (Recall that an involution is a permutation that equals its inverse.)

3. Let $A_d(x)$ denote the dth Eulerian polynomial. Show that every zero of $A_d(x)$ is real.
Hint: Recall the formula proved in class relating $A_{d+1}(x)$, $A'_d(x)$ and $A_d(x)$.

4. Using only combinatorial arguments and the definitions of $\sec x$ and $\tan x$ in terms of Euler numbers, prove that
$$\frac{d}{dx}\sec^2 x = 2\sec^2 x \tan x.$$

5. A cycle in a permutation is said to be *up-down* if, when written with its smallest element first, say $(b_1, b_2, \dots)$, we have that $b_1 < b_2 > b_3 < \dots$. Let Δ_n denote the set of permutations of $[n]$ that can be written as a product of up-down cycles. For example, $(1,5,2,7)(3)(4,8,6)(9) \in \Delta_9$, but $(1,3,5)(2,4)(6) \notin \Delta_6$. Prove that
$$|\Delta_n| = E_{n+1}.$$

6. Let $k < n/2$. Find a bijection f from the set of k-element subsets of $[n]$ to the set of $(n-k)$-element subsets of $[n]$ with the property that for every k-element subset S, $S \subseteq f(S)$.