Math 108. Topics in Combinatorics.

Problem Set 2. Due on Thursday, 10/13/16.

1. Find a closed formula for the sum

$$\sum_{\pi \in S_n} \mathrm{maj}(\pi).$$

2. Let $P_{n,k}$ be the 'slice' of the n-dimensional cube $0 \leq x_i \leq 1$, $i = 1, \ldots, n$, between the hyperplanes $\sum_{i=1}^{n} x_i = k$ and $\sum_{i=1}^{n} x_i = k+1$. Prove that

$$n! \operatorname{Vol}(P_{n,k}) = \#\{\pi \in S_n : \pi \text{ has } k \text{ descents}\}.$$

3. Recall Sylvester's map $\lambda \mapsto \mu$ defined in class (or see [EC1, Prop. 1.8.5, third proof]), where λ is a partition into odd parts.

   (a) Prove that the image μ is a partition into distinct parts.

   (b) Prove that Sylvester's map is a bijection between partitions of n into odd parts and partitions of n into distinct parts.

   (c) (Bonus) Prove that the number of different parts in λ equals the number of blocks of consecutive parts in μ. For example, $\lambda = (9, 9, 7, 3, 3)$ has three different parts, and $\mu = (9, 8, 7, 4, 2, 1)$ has three blocks, namely block $9, 8, 7$, block 4, and block $2, 1$.

4. Recall that $p(n)$ denotes the number of partitions of n. Prove that the number of pairs (λ, μ) where $\lambda \vdash n$, $\mu \vdash n+1$, and the Young diagram of μ is obtained from that of λ by adding one square, is equal to $p(0) + p(1) + \cdots + p(n)$.

5. Prove that the number of partitions of n into 4 parts equals the number of partitions of $3n$ into 4 parts of size at most $n-1$.

6. Let $e(n) = \#$ of partitions of n with an even number of even parts,
   $o(n) = \#$ of partitions of n with an odd number of even parts.
   Show that $e(n) - o(n) = \#$ of self-conjugate partitions of n.
   Recall that λ is self-conjugate if $\lambda = \lambda'$.