Mathematics 111

Spring 2007

Homework 4

1. (Extending the base). Let $\varphi : A \to B$ be a ring homomorphism. We can view B as a left and right A-module via $a \cdot b := \varphi(a)b$ and $b \cdot a := b\varphi(a)$. A standard example is where $A \subset B$ and φ is inclusion. So we can view B as an $A - A$ bi-module. Let N be a left A-module.

   (a) Show that the abelian group $B \otimes_A N$ has a left A-module structure and that the map $N \to B \otimes_A N$ given by $n \mapsto 1_B \otimes n$ is A-linear. Note that $B \otimes_A N$ has a natural left B-module structure, so it is natural to wonder to what extent $B \otimes_A N$ is a B-module "containing" the original A-module N. Not always as you will show, but it is true in many important cases.

   (b) First the bad news. Give an example where φ is injective, but the associated map $N \to B \otimes_A N$ is not.

   (c) Now suppose that N is a free A-module with basis $\{n_i\}_{i \in I}$. From class we know that $B \otimes_A N$ is a free left B-module with basis $\{1_B \otimes n_i\}_{i \in I}$. Under the assumption N is free, show that if φ is injective, so too is the map $N \to B \otimes_A N$. In particular, if L/K is a field extension and V is a vector space over K, then $L \otimes_K V$ is a vector space over L "containing" V and having the same dimension as V over K.

   (d) As an example of the above, show that if L/K is a field extension and $V = K[x]$ is the polynomial ring over K, there is a natural isomorphism (as L-modules) between $L[x]$ and $L \otimes_K K[x]$.

   (e) The previous isomorphism may be more subtle than it appears. Let K be a field and α an element in its algebraic closure. Despite the natural L-linear isomorphism between $L[x]$ and $L \otimes_K K[x]$, show that there are choices of L and α so that $L[\alpha]$ and $L \otimes_K K[\alpha]$ are not isomorphic as L-modules.

2. Some applications. Let K be a field, f an irreducible polynomial in $K[x]$, and α a root of f in some splitting field.

   (a) Show that for any field extension L/K, $L \otimes_K K(\alpha)$ is isomorphic (as an L-module) to the quotient $L[x]/(f(x))$.

   (b) Let $K = \mathbb{Q}$, $\alpha = \sqrt[6]{2}$, $L = \mathbb{Q}(\sqrt[3]{2})$. $L \otimes_K K(\alpha)$ is isomorphic to a direct sum of fields (each an extension of L). Find them.

   (c) Let L be a splitting field for a separable polynomial f over K, $\alpha \in L$ a root of f. Show that $L \otimes_K K(\alpha)$ is isomorphic to L^n where n is the degree of f.