

Mathematics 111
Spring 2011
Homework 5

1. Let R be a PID and M a finitely generated R -module. Show that M is projective if and only if it is free.
2. Let M be a submodule of \mathbb{Z}^n having group index p in M , i.e., $[\mathbb{Z}^n : M] = p$, where p is a prime. Show that M is free of rank n , and there is a basis $\{e_1, \dots, e_n\}$ of \mathbb{Z}^n so that $M = \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_{n-1} \oplus \mathbb{Z}pe_n$.
3. Show that a vector $v = (a_1, \dots, a_n) \in \mathbb{Z}^n$ extends to a basis $\{v, v_2, \dots, v_n\}$ of \mathbb{Z}^n if and only if the a_i are coprime, that is $a_1\mathbb{Z} + \dots + a_n\mathbb{Z} = \mathbb{Z}$. *Hint:* For one direction, come up with a short exact sequence that splits.
4. Let F_1 and F_2 be free modules of (not necessarily the same) finite rank over a PID R . Let $\varphi : F_1 \rightarrow F_2$ be R -linear and nontrivial. Show that there exists bases $\{v_1, \dots, v_n\}$ of F_1 and $\{w_1, \dots, w_m\}$ of F_2 together with elements a_1, \dots, a_r of R so that

$$\varphi(v_i) = \begin{cases} a_i w_i & 1 \leq i \leq r \\ 0 & r + 1 \leq i \leq n, \end{cases}$$

with $a_1 \mid a_2 \mid \dots \mid a_r$, and the ideals $a_j R$ uniquely determined.

5. Let $A = \begin{pmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{pmatrix}$.

- (a) If $\varphi : \mathbb{Z}^3 \rightarrow \mathbb{Z}^2$ is a \mathbb{Z} -linear map whose matrix with respect to the standard bases is A , determine the structure of the cokernel $\mathbb{Z}^2 / \text{Im}(\varphi)$ as a direct sum of cyclic groups. Find a minimal set of generators for the quotient. *Hint:* The image of φ is the span of the columns (i.e., the column space), and you may assume without loss of generality that elementary column operations (over \mathbb{Z}) leave the column space unchanged. Explain how your answer is connected to the invariant factor theorem.

- (b) Determine all integer solutions to $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$. *Hint:* Elementary row operations (over \mathbb{Z}) do not change the kernel.