Your Name Goes Here

Dartmouth College

Mathematics 81/111 — Homework 4

1. Let F be a field of characteristic 0, and let m and n be distinct integers with $\sqrt{m} \notin F$, $\sqrt{n} \notin F$, and $\sqrt{mn} \notin F$.
   (a) Show that $[F(\sqrt{m}, \sqrt{n}) : F] = 4$.
   (b) Show by example (with $m/n \notin (\mathbb{Q}^\times)^2$) that the above statement can be false if we only assume that $\sqrt{m} \notin F$ and $\sqrt{n} \notin F$.
   (c) Let $m_1, m_2, \ldots, m_t$ be square-free integers ($m_i \neq 0, \pm 1$) which are relatively prime in pairs. Show that $[\mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_t}) : \mathbb{Q}] = 2^t$. Hint: A *careful* induction on t may be of use.

2. Show that the class of algebraic field extensions is a distinguished class. Note that 'in theory' Lang has a proof of this in the text, but at least the second part is incredibly terse. I want nice detailed proofs.

   Recall that a class $\mathcal{C}$ of field extensions is *distinguished* if it satisfies three properties:

   I. Consider a tower of fields $K \subset F \subset E$. The extension E/K is in $\mathcal{C}$ if and only if E/F and F/K are in $\mathcal{C}$.
   II. If E/K is in $\mathcal{C}$, and F/K is any extension of K (and E, F lie in some common field), then EF/F is in $\mathcal{C}$.
   III. If E/K and F/K are in $\mathcal{C}$ (and E, F lie in some common field), then EF/K is in $\mathcal{C}$.

   We have shown that I and II imply III.

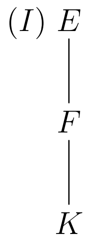
3. Field extensions.
   (a) Let $\alpha = \sqrt[11]{5} \in \mathbb{R}$. Determine (and justify) the degree of $\mathbb{Q}(\beta)/\mathbb{Q}$ where $\beta = 3 - 2\alpha + 4\alpha^4 - 5\alpha^9$.
   (b) Let $n \geq 3$, $\zeta_n = e^{2\pi i/n}$, and consider the cyclotomic field $K = \mathbb{Q}(\zeta_n)$, and $F = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$. Show that $F \subset \mathbb{R}$, and $[K : F] = 2$. The field F is called the maximal real subfield of K.
   (c) When $n = 5$ show that $[F : \mathbb{Q}] = 2$, and write $F = \mathbb{Q}(\sqrt{r})$ for some rational number r. Also write $\cos(2\pi/5)$ in terms of radicals of rational numbers.

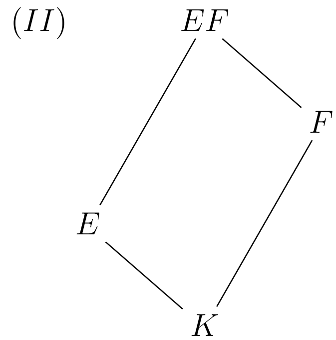
4. Let $m > 1$ be a square-free integer, and $n \geq 1$ an odd integer. Let $F/\mathbb{Q}$ be any field extension with $[F : \mathbb{Q}] = 2$. Show that $x^n - m$ is irreducible in $F[x]$.

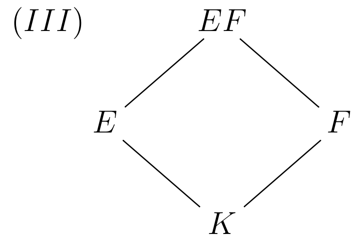
(problem 5 on next page)

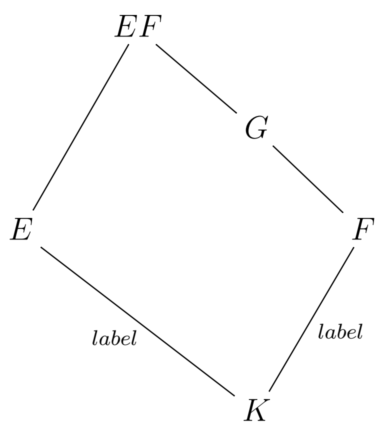
5. Determine the splitting field over $\mathbb{Q}$ (and its degree) of $x^4 + x^2 + 1$.

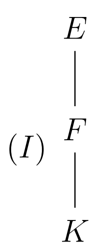
The following might also be useful diagrams for you LaTeXers. First via xy-pic, then via tikz.

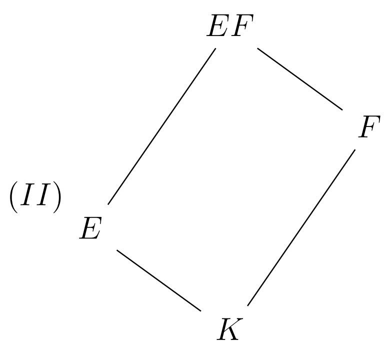


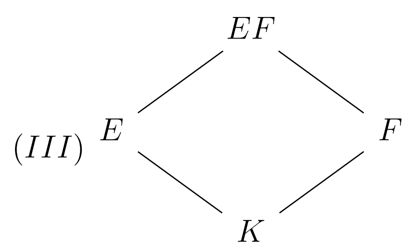


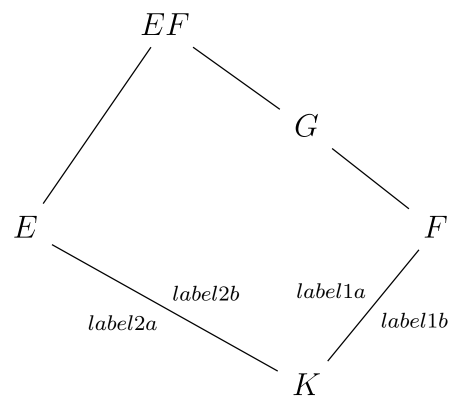








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