Math 112: GGT Fall 2015 - solution to exercise 16

We recall from the lecture:

**Theorem 1** Let $(X_1, d_1)$, $(X_2, d_2)$ and $(X_3, d_3)$ be metric spaces. If

$$f : X_1 \to X_2 \quad \text{and} \quad g : X_2 \to X_3$$

are quasi-isometries, then $g \circ f$ is also a quasi-isometry.

**Theorem 2** Let $(X_1, d_1)$ and $(X_2, d_2)$ be two metric spaces and $f, g : X_1 \to X_2$ be two functions. Then the relation $\sim$ given by

$$f \sim g \iff \sup_{x \in X_1} d_2(f(x), g(x)) < \infty$$

defines an equivalence relation.

We have to show:

a) $(\text{QIC}(X), \circ)$, the set of equivalence classes (with respect to $\sim$) of quasi-isometries from $X$ to $X$ forms a group.

b) A quasi-isometry $h : X \to Y$ induces a group isomorphism $\phi_h : \text{QIC}(X) \to \text{QIC}(Y)$.

**proof:** To prove a) we show:

1.) If $f : X \to X$ is a $(\lambda, c)$ quasi-isometry and $g \sim f$. Then $g$ is also a quasi-isometry.

2.) The composition $\circ$ preserves equivalence classes.

3.) $(\text{QIC}(X), \circ)$ is a group.

1.) Let $f : X \to X$ be a $(\lambda, c)$-quasi-isometry and $g \sim f$. We have to show that $g$ is also a quasi-isometry. As $g \sim f$ there is a $K \geq 0$, such that $d(f(x), g(x)) \leq K$ for all $x \in X$ and we have, as $f$ is a quasi-isometric embedding:

$$d(g(x), g(y)) \leq d(g(x), f(x)) + d(f(x), f(y)) + d(f(y), g(y)) \leq \lambda d(x, y) + c + 2K \quad \text{for} \quad x, y \in X.$$ 

Furthermore, for $x, y \in X$

$$\frac{1}{\lambda} d(x, y) - c \leq d(f(x), f(y)) \leq d(f(x), g(x)) + d(g(x), g(y)) + d(g(y), f(y)).$$

Using the same arguments as above we obtain

$$\frac{1}{\lambda} d(x, y) - c - 2K \leq d(g(x), g(y)).$$

Additionally $f$ is $D$ quasi-surjective. Therefore for any $x \in X$ there is a $y \in X$, such that $d(f(y), x) \leq D$, and it follows that

$$d(x, g(y)) \leq d(x, f(y)) + d(f(y), g(y)) \leq D + K.$$ 

Therefore $g$ is a $D + K$-quasi-surjective map and in total we obtain that $g$ is a quasi-isometry.

Note that by **Theorem 1** we have that if $f, g : X \to X$ are quasi-isometries, then $g \circ f$ is also a quasi-isometry.

2.) The composition $\circ$ preserves equivalence classes:

To show that the composition of maps descends to a composition on the set of equivalence classes,
we have to show that for given quasi-isometries \( f, f', g, g' \) on \( X \), where \( f \sim f' \) and \( g \sim g' \) we have that \( gf = g \circ f \sim g' \circ f' = g'f' \). Now \( g \) is a \((\lambda_g, c_g)\) quasi-isometry and

\[
\sup_{x \in X} d(f(x), f'(x)) \leq K_f, \quad \sup_{x \in X} d(g(x), g'(x)) \leq K_g
\]

for two constants \( K_f, K_g \geq 0 \). Hence for \( x \in X \) we have

\[
d(gf(x), g'f'(x)) \leq d(gf(x), g'(x)) + d(g'(x), g'(x))
\]

\[
\leq \lambda_g d(f(x), f'(x)) + c_g + K_g
\]

\[
\leq \lambda_g K_f + c_g + K_g.
\]

Therefore \( gf \sim g'f' \). In the following we write \([f]\) for the equivalence class of \( f \).

3.) \((\mathrm{QIC}(X), \circ)\) is a group:

By Theorem 1 we have that if \( f, g : X \to X \) are quasi-isometries, then \( g \circ f \) is also a quasi-isometry. Hence \( \mathrm{QIC}(X) \) is closed under the composition \( \circ \).

The associativity of the composition of maps transfers immediately to the composition on equivalence classes.

The identity is a quasi-isometry and its equivalence class is obviously the neutral element of \( \mathrm{QIC}(X) \). If \( f \) is a quasi-isometry, then we have seen in the lecture that there exists a quasi-inverse quasi-isometry \( g \). By the quasi-surjectivity of \( f \) and \( g \) we have that

\[
d(gf(x), x) \leq K \quad \text{and} \quad d(fg(x), x) \leq K
\]

for some constant \( K \geq 0 \).

But this means that \([gf] = [id]\) and \([fg] = [id]\), therefore every element has an inverse. Therefore \( \mathrm{QIC}(X) \) is a group. This concludes the proof of part a).

To prove part b) we have to show:

4.) The map \( \phi_h : \mathrm{QIC}(X) \to \mathrm{QIC}(Y), [f] \mapsto [hfh'] \), where \( h' \) is a quasi-inverse for \( h \) is a group isomorphism:

Let \( h : X \to Y \) be a \((\lambda_h, c_h)\) quasi-isometry.

We first note that \( \phi_h \) is a well-defined map, as \([f] \in \mathrm{QIC}(X)\), and \( f' \in [f] \), that means \( d_X(f'(x), f(x)) \leq K_1 \) for all \( x \in X \) and a constant \( K_1 \geq 0 \). We have for \( y \in Y \)

\[
d_Y(hfh'(y), h'fh'(y)) \leq \lambda_h d_X(fh'(y), f'h'(y)) + c_h \leq \lambda_h K_1 + c_h < \infty,
\]

therefore \( hfh' \sim h'fh' \).

\( \phi_h \) is a group homomorphism. For \([f], [g] \in \mathrm{QIC}(X)\) we have \( hfh'ggh' \sim hfggh' \). Because for all \( y \in Y \)

\[
d_Y(hfh'ggh'(y), hfggh'(y)) \leq \lambda_h d_X(h'ggh'(y), gh'(y)) + c_{hf} \leq \lambda_h K_2 + c_{hf},
\]

as \( h'f \) is a \((\lambda_{hf}, c_{hf})\) quasi-isometry (for given constants) and \( h'g \) is a quasi-identity on \( X \) (from which we obtain the constant \( K_2 \geq 0 \)).

Finally we have \( \psi_h : \mathrm{QIC}(Y) \to \mathrm{QIC}(X), [g] \mapsto \psi_h([g]) := [h'gh] \) is the inverse map of \( \phi_h \):

Clearly \( \psi_h \) is a well-defined homomorphism and it can be easily check that a quasi-isometry \( f : X \to X \) has bounded distance from \( h'fh'h' \), therefore

\[
\psi_h \circ \phi_h = \mathrm{id}_{\mathrm{QIC}(X)}.
\]

The equation \( \phi_h \circ \psi_h = \mathrm{id}_{\mathrm{QIC}(Y)} \) can then be obtained in a similar fashion. This concludes the proof of part b).