Math 112: Geometric Group Theory
Fall 2015 - Assignment 7

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keywords: ping-pong lemma, theorem of Milnor and Schwarz, PSL₂(Z)

exercise 19. (ping-pong lemma)
Let $G$ be a group operating on a set $X$, such that the following conditions are satisfied:

- For $k \geq 2$, let $H_1, H_2, \ldots, H_k$ be non-trivial subgroups of $G$, such that at least one of these subgroups contains more than two elements.
- $X$ contains $k$ pairwise disjoint, non-empty subsets $X_1, X_2, \ldots, X_k$, such that for all $i \neq j$ and all $h \in H_i \setminus \{1\}$ we have: $h(X_j) \subseteq X_i$.

Show that $\langle H_1 \cup H_2 \cup \ldots \cup H_k \rangle = H_1 * H_2 * \ldots * H_k$.

exercise 20. Consider the following two group actions and specify for each operation a condition of the theorem of Milnor and Schwarz that is fulfilled and one that is not fulfilled.

a) $SL_2(\mathbb{Z})$ operates on $\mathbb{R}^2$ via matrix multiplication.

b) $\mathbb{Z}$ operates on $X := \{(r^3, s) | r, s \in \mathbb{Z}\}$ (together with the induced Euclidean metric from $\mathbb{R}^2$) by

$$\phi : \mathbb{Z} \times X \to X, (n, (r^3, s)) \mapsto \phi(n, (r^3, s)) := (r^3, s + n).$$

exercise 21. (fundamental domain for the action of $PSL_2(\mathbb{Z})$ on $\mathbb{H}$ - part 1)
Let $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{-1\}$ be the group which we obtain by identifying $A \in SL_2(\mathbb{Z})$ with $-A$.

a) Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half-plane in $\mathbb{C}$. Let $\Gamma$ be the group of mappings

$$\Gamma := \{f_A : \mathbb{H} \to \mathbb{H} \mid f_A(z) = \frac{a \cdot z + c}{b \cdot z + d}, \text{ where } A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in PSL_2(\mathbb{Z})\}.$$ 

Show that $\phi(A)(\mathbb{H}) \subset \mathbb{H}$ for all $A \in PSL_2(\mathbb{Z})$ and that $\phi : PSL_2(\mathbb{Z}) \to \Gamma, A \mapsto \phi(A) := f_A$ is a group isomorphism.

b) Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Determine a fundamental domain $F_T$ of $\phi(\langle T \rangle)$ and $F_S$ of $\phi(\langle S \rangle)$. What are $\mathbb{H}\langle T \rangle$ and $\mathbb{H}\langle S \rangle$ topologically?

c) Show that: For all $z \in \mathbb{H}$ exists $A \in PSL_2(\mathbb{Z})$, such that $\text{Im}(\phi(A)(z))$ is maximal.

d) Determine a fundamental domain $F_{T,S}$ of $\phi(\langle T, S \rangle)$.